

# On $\frac{\pi}{2}$ -separated subsets of Alexandrov spaces with curvature $\geq 1$ <sup>1</sup>

*Dedicated to Xiaochun Rong for his 60th birthday*

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**Abstract.** Let  $M$  be an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ , and let  $\{q_1, \dots, q_k\}$  be any  $\frac{\pi}{2}$ -separated subset in  $M$  (i.e. the distance  $|q_i q_j| \geq \frac{\pi}{2}$  for any  $i \neq j$ ). Under the additional conditions “ $|q_i q_j| < \pi$ ” and “the diameter  $\text{diam}(M) \leq \frac{\pi}{2}$ ”, we respectively give the upper bound of  $k$  (which depends only on  $n$ ), and we classify the (topological or geometric) structure of  $M$  when  $k$  attains the upper bound.

**Key words.** Alexandrov spaces, positive curvature,  $\frac{\pi}{2}$ -separated subsets, rigidity

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## 0 Introduction

In studying the Morse theory on Alexandrov spaces with curvature  $\geq \kappa$  ([P1]), the following basic and easy idea plays an important role.

**Theorem 0.1.** *Let  $\{q_1, \dots, q_k\}$  be a subset in an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ . If the distance  $|q_i q_j| > \frac{\pi}{2}$  for any  $1 \leq i \neq j \leq k$ , then  $k \leq n + 2$ .*

However, Theorem 0.1 is not formulated in [P1] (maybe due to the simplicity of it, especially to Perelman). On the  $M$  in Theorem 0.1, in [GW] there are the following two packing radius theorems when  $k \geq 2$ : one is that  $M$  is homeomorphic to the join  $\mathbb{S}^{k-2} * N$ ; the other is that  $\min_{i \neq j} \{|q_i q_j|\} \leq \arccos(\frac{-1}{k-1})$ , and if the equality holds then  $M$  is isometric to the join  $\mathbb{S}^{k-2} * N$ , where  $\mathbb{S}^{k-2}$  is the  $(k-2)$ -dimensional unit sphere and  $N$  is some  $(n-k+1)$ -dimensional Alexandrov space with curvature  $\geq 1$ . (See the comments after Theorem B below for the definition of the join.)

We find that if “ $|q_i q_j| > \frac{\pi}{2}$ ” is changed to “ $|q_i q_j| \geq \frac{\pi}{2}$ ” in Theorem 0.1, to find the upper bounds of  $k$  under the condition

$$|q_i q_j| < \pi$$

is more interesting and difficult. In the present paper, we make clear this upper bound, and we can classify the geometric structure of  $M$  if  $k$  attains the upper bound. Note that the condition “ $|q_i q_j| > \frac{\pi}{2}$ ” in Theorem 0.1 implies that  $|q_i q_j| < \pi$  if  $k \geq 3$  (see Lemma 1.2 below), which is an important idea in [P1]. Hence, the condition “ $|q_i q_j| < \pi$ ” is not an artificial one.

Of course, if we only change “ $|q_i q_j| > \frac{\pi}{2}$ ” to “ $|q_i q_j| \geq \frac{\pi}{2}$ ” in Theorem 0.1, we have the following well-known result (cf. [GW]).

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**Theorem 0.2.** *Let  $\{q_1, \dots, q_k\}$  be a subset in an  $n$ -dimensional Alexandrov space with curvature  $\geq 1$ . If the distance  $|q_i q_j| \geq \frac{\pi}{2}$  for any  $1 \leq i \neq j \leq k$ , then  $k \leq 2(n+1)$ , and if the equality holds then  $M$  is isometric to  $\mathbb{S}^n$  and we can rearrange all  $q_i$  such that  $|q_{2j-1} q_{2j}| = \pi$  for  $1 \leq j \leq n+1$ .*

For the completeness of the paper, we will give a proof of Theorem 0.2 in Appendix.

In the present paper, we let  $\mathcal{A}^n(\kappa)$  denote the collection of all  $n$ -dimensional Alexandrov spaces with curvature  $\geq \kappa$  (containing all  $n$ -dimensional Riemannian manifolds with sectional curvature  $\geq \kappa$ ), and without special remark we always consider complete spaces in  $\mathcal{A}^n(\kappa)$ .

**Definition 0.3.** Let  $M \in \mathcal{A}^n(1)$ , and let  $Q \triangleq \{q_1, \dots, q_k\}$  be a subset in  $M$ . We call  $Q$  a  $\frac{\pi}{2}$ -separated subset in  $M$  if the distance  $|q_i q_j| \geq \frac{\pi}{2}$  for any  $1 \leq i \neq j \leq k$ .

Now we give our first estimate result.

**Theorem A** *Let  $M \in \mathcal{A}^n(1)$ , and let  $\{q_1, \dots, q_k\}$  be a  $\frac{\pi}{2}$ -separated subset in  $M$ . If  $|q_1 q_i| > \frac{\pi}{2}$  for any  $2 \leq i \leq k$ , then  $k \leq n+2$ ; and if the equality holds, then  $M$  is homeomorphic to  $\mathbb{S}^n$  (and thus  $M$  has empty boundary).*

Note that Theorem A implies Theorem 0.1. Since the idea of estimating  $k$  in Theorem A is the same as in Theorem 0.1, the upper bound of  $k$  in Theorem A should be known to experts. For the convenience of readers, we will give its proof in Section 1. However, the following results are not so obvious.

**Theorem B** *Let  $M \in \mathcal{A}^n(1)$ , and let  $\{q_1, \dots, q_k\}$  be a  $\frac{\pi}{2}$ -separated subset in  $M$ . If  $|q_i q_j| < \pi$  for any  $1 \leq i \neq j \leq k$ , then*

$$k \leq 3l \text{ (resp. } 3l+1 \text{) for } n = 2l-1 \text{ (resp. } 2l\text{);}$$

*moreover, if the equality holds, then we can rearrange all  $q_i$  such that  $M$  is isometric to  $S_1^1 * \dots * S_l^1$  (resp. if  $M$  has empty boundary, then either  $M$  is isometric to  $S_1^1 * \dots * S_{l-1}^1 * N$  for some  $N \in \mathcal{A}^2(1)$ , or  $\{q_{3l+1}\} * S_1^1 * \dots * S_l^1$  can be isometrically embedded into  $M$ ; if  $M$  has nonempty boundary, then  $M$  is isometric to  $\{q_{3l+1}\} * S_1^1 * \dots * S_l^1$ ) with  $S_j^1$  having perimeter  $\geq \frac{3\pi}{2}$  (of course  $\leq 2\pi$ ) and  $q_{3j-2}, q_{3j-1}, q_{3j} \in S_j^1$  for each  $j$ .*

Recall that a join  $X * Y$  with  $X, Y \in \mathcal{A}(1)$  is defined as follows ([BGP]).  $X * Y = X \times Y \times [0, \frac{\pi}{2}] / \sim$ , where  $(x, y, t) \sim (x', y', t') \Leftrightarrow t = t' = 0$  and  $x = x'$  or  $t = t' = \frac{\pi}{2}$  and  $y = y'$ , and for any  $p_i = [(x_i, y_i, t_i)] \in X * Y$

$$\cos |p_1 p_2| = \cos t_1 \cos t_2 \cos |x_1 x_2| + \sin t_1 \sin t_2 \cos |y_1 y_2|.$$

Note that  $X * Y$  also belongs to  $\mathcal{A}(1)$  and  $\dim(X * Y) = \dim(X) + \dim(Y) + 1$ , and  $X * Y$  is a Riemannian manifold if and only if  $X$  and  $Y$  are isometric to unit spheres.

A very interesting corollary of Theorem B is on Riemannian cases.

**Corollary C** *Let  $M$  be a closed  $n$ -dimensional Riemannian manifold with sectional curvature  $\geq 1$ , and let  $\{q_1, \dots, q_k\}$  be a  $\frac{\pi}{2}$ -separated subset in  $M$  with  $|q_i q_j| < \pi$  for*

any  $1 \leq i \neq j \leq k$ . Then  $k \leq 3l$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ); and if the equality holds and  $n > 2$ , then  $M$  is isometric to the unit sphere  $\mathbb{S}^n$ .

When  $n = 3$ , for example, Corollary C says that  $M$  contains at most 6 points  $q_1, \dots, q_6$  with  $\frac{\pi}{2} \leq |q_i q_j| < \pi$  for any  $1 \leq i \neq j \leq 6$ , and only the unit sphere  $\mathbb{S}^3$  contains such 6 points (if we embed  $\mathbb{S}^3$  isometrically into the Euclidean space  $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) | x_i \in \mathbb{R}\}$ , we can select the former (resp. latter) 3 points on the plane  $\{(x_1, x_2, 0, 0)\}$  (resp.  $\{(0, 0, x_3, x_4)\}$ ). And this is the unique way to select such 6 points up to an orthogonal transformation of  $\mathbb{R}^4$ ).

**Theorem D** *Let  $M \in \mathcal{A}^n(1)$ , and let  $\{q_1, \dots, q_k\}$  be a  $\frac{\pi}{2}$ -separated subset in  $M$ . If the diameter  $\text{diam}(M) \leq \frac{\pi}{2}$ , then there exists an isometrical embedding  $f : \Delta_+^{k-1} \rightarrow M$  such that  $q_1, \dots, q_k$  are the vertices of  $f(\Delta_+^{k-1})$ , where*

$$\Delta_+^{k-1} \triangleq \left\{ (x_1, \dots, x_k) \in \mathbb{R}^k \mid \sum_{i=1}^k x_i^2 = 1, x_i \geq 0 \right\} \subset \mathbb{S}^{k-1}.$$

As a result,  $k \leq n + 1$ ; moreover, if  $k = n + 1$ , then  $M$  is a glued space of finite copies of  $\Delta_+^n$  along some “faces”  $\Delta_+^{n-1}$  of them.

We know that the boundary of a  $\Delta_+^n$  consists of  $n + 1$  copies of  $\Delta_+^{n-1}$ . Here, such a  $\Delta_+^{n-1}$  is said to be a “face” of the  $\Delta_+^n$ .

Similarly, Theorem D has the following corollary on Riemannian manifolds.

**Corollary E** *If in addition  $M$  is a closed  $n$ -dimensional Riemannian manifold with sectional curvature  $\geq 1$  in Theorem D, and if  $k = n + 1$ , then  $M$  is isometric to the projective space  $\mathbb{RP}^n$  with the canonical metric (i.e. the metric induced from  $\mathbb{S}^n$ ).*

On Theorem D, we supply another Riemannian example (for more general examples please refer to Remark 3.12). We consider the complex projective space  $\mathbb{CP}^n$  with the canonical metric (i.e. the metric induced from  $\mathbb{S}^{2n+1}$ ). It is well known that  $\mathbb{CP}^n$  has sectional curvature  $\geq 1$  (and  $\leq 4$ ) and the diameter  $\leq \frac{\pi}{2}$ . By the induction on  $n$ , it is not hard to see that  $\mathbb{CP}^n$  contains  $\{q_1, \dots, q_{n+1}\}$  with  $|q_i q_j| = \frac{\pi}{2}$  for any  $1 \leq i \neq j \leq n + 1$  (however, we cannot find a  $\frac{\pi}{2}$ -separated subset containing  $n + 2$  points in  $\mathbb{CP}^n$ ). According to Theorem D,  $\Delta_+^n$  can be isometrically embedded into  $\mathbb{CP}^n$ .

We will end this section by introducing our main tool—the Toponogov Comparison Theorem, which is the essential geometry in Alexandrov spaces with curvature  $\geq \kappa$ .

We always let  $[pq]$  denote a geodesic (i.e. a shortest path) between  $p$  and  $q$  in  $M \in \mathcal{A}^n(\kappa)$ , and let  $\uparrow_p^q$  denote the direction at  $p$  of the geodesic  $[pq]$ . Given another geodesic  $[pr]$  in  $M$ , we let  $\angle qpr$  denote the angle between  $[pq]$  and  $[pr]$  at  $p$  (for the detailed contents of angles please refer to [BGP]). We know that  $\angle qpr$  is equal to  $|\uparrow_p^q \uparrow_p^r|$  (i.e. the distance between  $\uparrow_p^q$  and  $\uparrow_p^r$  in  $\Sigma_p M$ , where  $\Sigma_p M \in \mathcal{A}^{n-1}(1)$  is the direction space of  $M$  at  $p$ ).

We say that the geodesics  $[pq]$  and  $[pr]$  form a *hinge*  $p \prec_r^q$  at  $p$  with angle  $\angle qpr$ , and call an associated hinge  $\tilde{p} \prec_{\tilde{r}}^{\tilde{q}}$  in  $\mathbb{S}_{\kappa}^2$  with  $|\tilde{p}\tilde{q}| = |pq|$ ,  $|\tilde{p}\tilde{r}| = |pr|$  and  $\angle \tilde{q}\tilde{p}\tilde{r} = \angle qpr$  the

comparison hinge of  $p \prec_r^q$ , where  $\mathbb{S}_\kappa^2$  is the complete and simply-connected 2-manifold of constant curvature  $\kappa$ . Similarly, we say that geodesics  $[pq], [qr]$  and  $[rp]$  form a triangle  $\triangle pqr$ , and call an associated triangle  $\triangle \tilde{p}\tilde{q}\tilde{r}$  in  $\mathbb{S}_\kappa^2$  with  $|\tilde{p}\tilde{q}| = |pq|$ ,  $|\tilde{p}\tilde{r}| = |pr|$  and  $|\tilde{r}\tilde{q}| = |rq|$  the comparison triangle of  $\triangle pqr$ .

For any triangle  $\triangle pqr$  (we only need to consider the case  $|pq| + |pr| + |qr| < 2\pi/\sqrt{\kappa}$  if  $\kappa > 0$  ([BGP])) and hinge  $p \prec_r^q$  in  $M \in \mathcal{A}^n(\kappa)$  and their comparison triangle and hinge  $\triangle \tilde{p}\tilde{q}\tilde{r}$  and  $\tilde{p} \prec_{\tilde{r}}^{\tilde{q}}$ , the Toponogov Comparison Theorem (TCT) asserts that ([BGP]):

**Theorem 0.4** (TCT). (i) *For any two points  $s \in [qr] \subset \triangle pqr$  and  $\tilde{s} \in [\tilde{q}\tilde{r}] \subset \triangle \tilde{p}\tilde{q}\tilde{r}$  with  $|qs| = |\tilde{q}\tilde{s}|$ , we have  $|ps| \geq |\tilde{p}\tilde{s}|$ .*

(ii) *In  $\triangle pqr$  and  $\triangle \tilde{p}\tilde{q}\tilde{r}$ , we have  $\angle pqr \geq \angle \tilde{p}\tilde{q}\tilde{r}$ ,  $\angle qrp \geq \angle \tilde{q}\tilde{r}\tilde{p}$  and  $\angle rpq \geq \angle \tilde{r}\tilde{p}\tilde{q}$ .*

(iii) *In  $p \prec_r^q$  and  $\tilde{p} \prec_{\tilde{r}}^{\tilde{q}}$ , we have  $|\tilde{q}\tilde{r}| \geq |qr|$ .*

It is known that (i)-(iii) of TCT are equivalent to each other. Moreover, we have the following result when the “=” holds in TCT.

**Theorem 0.5** (TCT for “=” ([GM])). (i) *If there is a point  $s \in [qr]^\circ$  such that  $|ps| = |\tilde{p}\tilde{s}|$  in (i) of TCT, then for any given geodesic  $[ps]$  there exist unique two geodesics  $[pq]'$  and  $[pr]'$  (maybe not  $[pq]$  and  $[pr]$ ) such that the triangle formed by  $[pq]'$ ,  $[pr]'$  and  $[qr]$  is isometric to its comparison triangle.*

(ii) *If  $|\tilde{q}\tilde{r}| = |qr|$  in (iii) of TCT (or if  $\angle rpq = \angle \tilde{r}\tilde{p}\tilde{q}$  in (ii) of TCT), then there exists geodesic  $[qr]'$  (maybe not  $[qr]$ ) such that the triangle formed by  $[pq]$ ,  $[pr]$  and  $[qr]'$  is isometric to its comparison triangle.*

The rest of the paper is organized as follows. In Sections 1-3, we give the proofs of Theorem A, Theorem B and Corollary C, and Theorem D and Corollary E respectively. A technical corollary of Theorem D is given in Section 4. In Appendix, we will prove Theorem 0.2 and Lemmas 1.4 and 2.11.

## 1 Proof of Theorem A

In the paper, we often use the following lemma, an obvious corollary of Theorem 0.4.

**Lemma 1.1.** *Let  $M \in \mathcal{A}^n(1)$ , and let  $p, q, r \in M$  with  $|qr| \geq \frac{\pi}{2}$ . If either  $|pq|, |pr| \leq \frac{\pi}{2}$  or  $\frac{\pi}{2} \leq |pq|, |pr| < \pi$ , then for any geodesics  $[pq]$  and  $[pr]$  we have  $|\uparrow_p^q \uparrow_p^r| \geq \frac{\pi}{2}$  in  $\Sigma_p M$ ; and if in addition  $|qr| > \frac{\pi}{2}$  or  $|pq|, |pr| > \frac{\pi}{2}$ , then  $|\uparrow_p^q \uparrow_p^r| > \frac{\pi}{2}$ .*

And the following basic fact will be used sometimes.

**Lemma 1.2** ([BGP]). *Let  $M \in \mathcal{A}^n(1)$  and  $p, q \in M$ . If  $|pq| = \pi$ , then  $|px| + |qx| = \pi$  for any  $x \in M$ , and  $M = \{p, q\} * M_1$  for some  $M_1 \in \mathcal{A}^{n-1}(1)$ .*

Let  $M \in \mathcal{A}^n(1)$ . For  $n = 0$  and 1, we make the following convention: if  $n = 0$ , then  $M$  consists of one point or two points with distance equal to  $\pi$ ; if  $n = 1$ , then  $M$  is an arc with length  $\leq \pi$  or a circle with perimeter  $\leq 2\pi$ .

*Proof of Theorem A.*

We will give the proof by the induction on the dimension  $n$ . Obviously, Theorem A is true if  $n = 0$  and  $1$  (see the above convention). Now we assume that  $n > 1$ , and we can assume that  $k \geq 3$ . According to Lemma 1.2, “ $|q_1 q_i| > \frac{\pi}{2}$  for any  $2 \leq i \leq k$ ” implies that  $|q_i q_j| < \pi$  for any  $1 \leq i \neq j \leq k$ . Then by Lemma 1.1, any  $\{\uparrow_{q_k}^{q_1}, \dots, \uparrow_{q_k}^{q_{k-1}}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_k} M \in \mathcal{A}^{n-1}(1)$  with  $|\uparrow_{q_k}^{q_1} \uparrow_{q_k}^{q_i}| > \frac{\pi}{2}$  for all  $2 \leq i \leq k-1$ . By the inductive assumption on  $\Sigma_{q_k} M$ , we have

$$k-1 \leq n-1+2, \text{ i.e., } k \leq n+2.$$

Now we will prove that  $M$  is homeomorphic to  $\mathbb{S}^n$  if  $k = n+2$ . By the Radius Sphere Theorem ([GP]), it suffices to show that  $\text{rad}(M) > \frac{\pi}{2}$ , where  $\text{rad}(M)$  is the radius of  $M$  defined by  $\min_{p \in M} \{\max_{q \in M} |pq|\}$ . Note that if  $\text{rad}(M) \leq \frac{\pi}{2}$ , then there is a point  $p \in M$  such that  $|px| \leq \frac{\pi}{2}$  for all  $x \in M$ . Obviously,  $p \notin \{q_1, \dots, q_k\}$  (note that  $|q_1 q_i| > \frac{\pi}{2}$  for all  $2 \leq i \leq k$ ). Then by Lemma 1.1, any  $\{\uparrow_p^{q_1}, \dots, \uparrow_p^{q_k}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_p M \in \mathcal{A}^{n-1}(1)$  with  $|\uparrow_p^{q_1} \uparrow_p^{q_i}| > \frac{\pi}{2}$  for all  $2 \leq i \leq k$ , so by the former part (we have proved) we have  $k \leq n-1+2 = n+1$ ; a contradiction. It therefore has to hold that the radius  $\text{rad}(M) > \frac{\pi}{2}$  (and thus  $M$  is homeomorphic to  $\mathbb{S}^n$ ).  $\square$

In the above proof, we use the Radius Sphere Theorem to show that  $M$  is homeomorphic to  $\mathbb{S}^n$ , and thus  $M$  has empty boundary. In fact, we can prove that  $M$  has empty boundary (when  $k = n+2$  in Theorem A) without the Radius Sphere Theorem as follows.

**Proof 1.3** (a proof for “ $M$  has empty boundary if  $k = n+2$  in Theorem A”).

Obviously, this is true when  $n = 1$ . Next, we will derive a contradiction by applying the induction on  $n$  and assuming that the boundary  $\partial M \neq \emptyset$ .

We consider  $\Sigma_{q_2} M (\in \mathcal{A}^{n-1}(1))$ . From the above proof, we know that  $|q_i q_j| < \pi$  for any  $1 \leq i \neq j \leq n+2$ . By Lemma 1.1, any  $\{\uparrow_{q_2}^{q_1}, \uparrow_{q_2}^{q_3}, \dots, \uparrow_{q_2}^{q_{n+2}}\}$  is a  $\frac{\pi}{2}$ -separated subset with  $|\uparrow_{q_2}^{q_1} \uparrow_{q_2}^{q_i}| > \frac{\pi}{2}$  for any  $3 \leq i \leq n+2$ . By the inductive assumption,  $\Sigma_{q_2} M$  has empty boundary, so  $q_2 \notin \partial M$  (for the detailed contents on the boundary of a space in  $\mathcal{A}^n(\kappa)$  please refer to [BGP]). Then we select  $p \in \partial M$  such that  $|q_2 p| = |q_2 \partial M|$ . If  $|q_2 p| \geq \frac{\pi}{2}$ , then by Lemma 1.4 below  $M = \{q_2\} * \partial M$ , which contradicts “ $|q_2 q_1| > \frac{\pi}{2}$ ”. Now we can assume that  $|q_2 p| < \frac{\pi}{2}$ . On the other hand, since  $|q_2 p| \leq |q_2 x|$  for all  $x \in \partial M$ , by the first variation formula ([BGP]) we have

$$|\uparrow_p^{q_2} \xi| \geq \frac{\pi}{2}$$

for any geodesic  $[pq_2]$  and  $\xi \in \partial(\Sigma_p M)$  (refer to [BGP] for  $\partial(\Sigma_p M)$ ). By Lemma 1.4 below,

$$\Sigma_p M = \{\uparrow_p^{q_2}\} * \partial(\Sigma_p M).$$

Hence, for any geodesic  $[pq_i]$  with  $i \neq 2$ , we have  $|\uparrow_p^{q_2} \uparrow_p^{q_i}| \leq \frac{\pi}{2}$ . Due to Theorem 0.4, we can conclude that

$$|pq_i| \geq \frac{\pi}{2} \quad (i \neq 2)$$

by considering the comparison triangle of  $\triangle q_2 p q_i$  containing sides  $[p q_2]$  and  $[p q_i]$  (note that  $|p q_2| < \frac{\pi}{2}$ ,  $|q_2 q_i| \geq \frac{\pi}{2}$  and  $\angle q_2 p q_i \leq \frac{\pi}{2}$ ). Furthermore, by Lemma 1.2 again, we can conclude that  $|p q_i| < \pi$  for  $i \neq 2$  (because  $|q_1 q_j| > \frac{\pi}{2}$  for  $j = 3, \dots, n+2$ ). Hence, by Lemma 1.1, any  $\{\uparrow_p^{q_1}, \uparrow_p^{q_3}, \dots, \uparrow_p^{q_{n+2}}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_p M \in \mathcal{A}^{n-1}(1)$  with  $|\uparrow_p^{q_1} \uparrow_p^{q_j}| > \frac{\pi}{2}$  for any  $3 \leq j \leq n+2$ . By the inductive assumption,  $\Sigma_p M$  has empty boundary, which contradicts “ $p \in \partial M$ ”.  $\square$

**Lemma 1.4.** *Let  $M \in \mathcal{A}^n(1)$  with nonempty boundary. If  $|p \partial M| \geq \frac{\pi}{2}$  for some  $p \in M$ , then  $M = \{p\} * \partial M$ .*

It is easy to see that Lemma 1.4 is a corollary of the Doubling Theorem by Perelman ([P2]). For the convenience of readers, we will give an elementary proof for it in Appendix.

## 2 Proofs of Theorem B and Corollary C

We will prove the following generalized version of Theorem B.

**Theorem 2.1.** *Let  $M \in \mathcal{A}^n(1)$ , and let  $\{q_1, \dots, q_h, q_{h+1}, \dots, q_k\}$  be a  $\frac{\pi}{2}$ -separated subset in  $M$ . Suppose that  $|q_i q_j| < \pi$  for any  $1 \leq i \neq j \leq k$ , and that  $|q_1 q_i| > \frac{\pi}{2}$  for any  $2 \leq i \leq h$ . Then  $h \leq n+2$ , and the following hold:*

- (i) *If  $h = n+2$ , then  $k = n+2$ , and  $M$  has empty boundary.*
- (ii) *If  $h = n+1$ , then  $k \leq n+2$ ; and if the equality holds, then either  $M$  has empty boundary, or  $M = \{q_{n+2}\} * N$  for some  $N \in \mathcal{A}^{n-1}(1)$  without boundary.*
- (iii) *If  $4 \leq h \leq n$ , then  $k - h \leq 3l$  (resp.  $3l+1$ ) for  $n - h + 1 = 2l - 1$  (resp.  $2l$ ); and if the equality holds, then  $M$  is isometric to  $L * S_1^1 * \dots * S_l^1$  (resp. either  $M$  is isometric to  $N * S_1^1 * \dots * S_{l-1}^1$ , or  $L * S_1^1 * \dots * S_l^1 * \{q_i\}$  for some  $i > h$  can be isometrically embedded into  $M$ ), where  $S_j^1$  is of perimeter  $\geq \frac{3\pi}{2}$ ,  $L \in \mathcal{A}^{h-2}(1)$  and  $N \in \mathcal{A}^{h+1}(1)$ .*
- (iv) *If  $h \leq 3$ , then  $k \leq 3l$  (resp.  $3l+1$ ) for  $n = 2l - 1$  (resp.  $2l$ ); moreover, if the equality holds, then we can rearrange all  $q_i$  such that  $M$  is isometric to  $S_1^1 * \dots * S_l^1$  (resp. if  $M$  has empty boundary, then either  $M$  is isometric to  $S_1^1 * \dots * S_{l-1}^1 * N$  for some  $N \in \mathcal{A}^2(1)$ , or  $\{q_{3l+1}\} * S_1^1 * \dots * S_l^1$  can be isometrically embedded into  $M$ ; if  $M$  has nonempty boundary, then  $M$  is isometric to  $\{q_{3l+1}\} * S_1^1 * \dots * S_l^1$ ) with  $S_j^1$  having perimeter  $\geq \frac{3\pi}{2}$  (of course  $\leq 2\pi$ ) and  $q_{3j-2}, q_{3j-1}, q_{3j} \in S_j^1$  for each  $j$ .*

Obviously, the conclusion “ $h \leq n+2$ ” in Theorem 2.1 is included in Theorem A. And note that Theorem B is included in (iv) of Theorem 2.1.

In the following we will first give the proofs of (i) and (ii) in Theorem 2.1.

### Proof of (i) in Theorem 2.1:

By Theorem A,  $M$  has empty boundary, so we only need to show that  $k = n+2$ . If  $k > n+2$ , then we consider  $\Sigma_{q_{h+1}} M \in \mathcal{A}^{n-1}(1)$ . By Lemma 1.1, any  $\{\uparrow_{q_{h+1}}^{q_1}, \dots, \uparrow_{q_{h+1}}^{q_h}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_{h+1}} M$  with  $|\uparrow_{q_{h+1}}^{q_1} \uparrow_{q_{h+1}}^{q_i}| > \frac{\pi}{2}$  for any  $2 \leq i \leq h$ . By Theorem A, we have  $h \leq n-1+2$ , which contradicts  $h = n+2$ .

□

**Proof of (ii) in Theorem 2.1:**

Obviously, this is true if  $n = 0$  and  $1$ . Then we assume that  $n \geq 2$ , which implies that  $h \geq 3$ .

We first prove that  $k \leq n + 2$ . If  $k > n + 2$ , then in  $\Sigma_{q_{h+2}}M \in \mathcal{A}^{n-1}(1)$  any  $\{\uparrow_{q_{h+2}}^{q_1}, \dots, \uparrow_{q_{h+2}}^{q_h}, \uparrow_{q_{h+2}}^{q_{h+1}}\}$  is a  $\frac{\pi}{2}$ -separated subset with  $|\uparrow_{q_{h+2}}^{q_1} \uparrow_{q_{h+2}}^{q_i}| > \frac{\pi}{2}$  for any  $2 \leq i \leq h$  (by Lemma 1.1). Furthermore, by Lemma 1.2 we can conclude that  $|\uparrow_{q_{h+2}}^{q_i} \uparrow_{q_{h+2}}^{q_j}| < \pi$  for any  $1 \leq i \neq j \leq h + 1$ . Therefore, the  $\frac{\pi}{2}$ -separated subset  $\{\uparrow_{q_{h+2}}^{q_1}, \dots, \uparrow_{q_{h+2}}^{q_h}, \uparrow_{q_{h+2}}^{q_{h+1}}\}$  in  $\Sigma_{q_{h+2}}M \in \mathcal{A}^{n-1}(1)$  satisfies the conditions of Theorem 2.1. Then by (i) of Theorem 2.1, it has to hold that  $h < (n - 1) + 2 = n + 1$  which contradicts  $h = n + 1$ .

Next we only need to prove that if  $k = n + 2$  and if  $M$  has nonempty boundary, then  $M = \{q_{n+2}\} * \partial M$ . By Lemma 1.4, it suffices to show that  $|q_{n+2}\partial M| \geq \frac{\pi}{2}$ . If  $|q_{n+2}\partial M| < \frac{\pi}{2}$ , we select  $p \in \partial M$  such that  $|q_{n+2}p| = |q_{n+2}\partial M|$ . Then like Proof 1.3, we can get that  $\frac{\pi}{2} \leq |pq_i| < \pi$  for  $1 \leq i \leq n + 1$ . Hence, by Lemma 1.1 any  $\{\uparrow_p^{q_1}, \dots, \uparrow_p^{q_{n+1}}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_pM \in \mathcal{A}^{n-1}(1)$  with  $|\uparrow_p^{q_1} \uparrow_p^{q_i}| > \frac{\pi}{2}$  for any  $2 \leq i \leq n + 1$ . By (i) of Theorem 2.1,  $\Sigma_pM$  has empty boundary, which contradicts  $p \in \partial M$ . □

## 2.1 Some preparations for proving (iii) and (iv) of Theorem 2.1

Given a subset  $A$  of  $M$ , we let  $A^{\geq d} \triangleq \{x \in M \mid |xa| \geq d, \forall a \in A\}$ . And similarly we can define the corresponding  $A^{\leq d}$ ,  $A^{=d}$ ,  $A^{< d}$  and  $A^{> d}$ . From Theorem 0.4, we can immediately see the following lemma.

**Lemma 2.2.** *Let  $A$  be a subset of  $M \in \mathcal{A}^n(1)$ . Then  $A^{\geq \frac{\pi}{2}}$  is convex in  $M$ .*

Recall that  $N$  is said to be *convex* in  $M \in \mathcal{A}^n(1)$  if there is a geodesic  $[xy]$  belonging to  $N$  for any  $x, y \in N$ , or  $N$  consists of two points with distance equal to  $\pi$ , or  $N$  consists of only one point. We know that a convex subset  $N$  in  $M$  also belongs to  $\mathcal{A}^m(1)$ ; and if  $N \subsetneq M$  and  $N$  has empty boundary, then  $m < n$  ([BGP]).

**Lemma 2.3.** ([Y]) *Let  $M \in \mathcal{A}^n(1)$ , and let  $A$  be a complete locally convex subset in  $M$ . If  $A$  has empty boundary, then  $A^{\geq \frac{\pi}{2}} = A^{=\frac{\pi}{2}}$ .*

In our proof, we will use a special and generalized case of Lemma 2.3.

**Lemma 2.4.** *Let  $M \in \mathcal{A}^n(1)$ , and let  $A \triangleq \bigcup_{i=0}^l [p_i p_{i+1}] \subset M$  (where  $p_{l+1} = p_0$ ). If geodesics  $\{[p_i p_{i+1}]\}_{i=0}^l$  satisfy  $|\uparrow_{p_i}^{p_{i-1}} \uparrow_{p_i}^{p_{i+1}}| = \pi$  (where  $p_{-1} = p_l$ ), then  $A^{\geq \frac{\pi}{2}} = A^{=\frac{\pi}{2}}$ .*

From Lemma 2.5 below, we have that  $\dim(A) + \dim(A^{=\frac{\pi}{2}}) \leq n - 1$  in Lemmas 2.3 and 2.4.

**Lemma 2.5.** ([RW]) *Let  $M \in \mathcal{A}^n(1)$ , and let  $N_1$  and  $N_2$  be two locally convex subsets in  $M$ . If  $|x_1 x_2| = \frac{\pi}{2}$  for all  $x_i \in N_i$ , then  $\dim(N_1) + \dim(N_2) \leq n - 1$ .*

Based on Lemmas 2.2-2.5, we will give the proofs of (iii) and (iv) in Theorem 2.1 by the induction on  $n$ . Because the inductive processes of the proofs for (iii) and (iv) are almost identical (please see Remark 2.10 below for the main difference between them), we only give the detailed proof for (iv). Obviously, (iv) is true when  $n = 1$ .

## 2.2 Estimating $k$ for $n = 2$ and 3 in (iv) of Theorem 2.1

In this subsection, we will mainly prove that  $k \leq 4$  and 6 when  $n = 2$  and 3 respectively in (iv) of Theorem 2.1. In the proof, we need the following technique lemma.

**Lemma 2.6.** *Let  $\{q_1, q_2, q_3\}$  be a  $\frac{\pi}{2}$ -separated subset in  $M \in \mathcal{A}^n(1)$ . If  $|q_1 q_i| < \pi$  and there are geodesics  $[q_1 q_i]$  ( $i = 2, 3$ ) such that  $|\uparrow_{q_1}^{q_2} \uparrow_{q_1}^{q_3}| = \pi$ , then  $\{q_1, q_2, q_3\}^{\geq \frac{\pi}{2}} \subseteq \{q_2, q_3\}^{=\frac{\pi}{2}}$ ; and for any  $z \in \{q_1, q_2, q_3\}^{\geq \frac{\pi}{2}} \cap \{q_1\}^{< \pi}$ , there is a geodesic  $[q_i z]$  such that  $\frac{\pi}{2} \leq |\uparrow_{q_i}^z \uparrow_{q_i}^{q_1}| < \pi$  ( $i = 2, 3$ ).*

*Proof.* We first note that if  $z \in \{q_1, q_2, q_3\}^{\geq \frac{\pi}{2}} \cap \{q_1\}^{=\pi}$ , then  $z \in \{q_2, q_3\}^{=\frac{\pi}{2}}$  by Lemma 1.2. Now let  $z$  be an arbitrary point in  $\{q_1, q_2, q_3\}^{\geq \frac{\pi}{2}} \cap \{q_1\}^{< \pi}$ . For any geodesic  $[q_1 z]$ , by Lemma 1.1 we have  $|\uparrow_{q_1}^z \uparrow_{q_1}^{q_2}| \geq \frac{\pi}{2}$  and  $|\uparrow_{q_1}^z \uparrow_{q_1}^{q_3}| \geq \frac{\pi}{2}$ . Together with the condition  $|\uparrow_{q_1}^{q_2} \uparrow_{q_1}^{q_3}| = \pi$ , this implies that

$$|\uparrow_{q_1}^z \uparrow_{q_1}^{q_2}| = |\uparrow_{q_1}^z \uparrow_{q_1}^{q_3}| = \frac{\pi}{2}.$$

Then by applying Theorem 0.4 on any triangle  $\triangle q_1 q_i z$  ( $i = 2, 3$ ), it is not hard to see that  $|q_i z| = \frac{\pi}{2}$  (i.e.  $z \in \{q_2, q_3\}^{=\frac{\pi}{2}}$ ); and then by Theorem 0.5, there is a geodesic  $[q_i z]$  such that the triangle  $\triangle q_1 q_i z$  formed by  $[q_1 q_i]$ ,  $[q_1 z]$  and  $[q_i z]$  is isometric to its comparison triangle, which implies that  $\frac{\pi}{2} \leq |\uparrow_{q_i}^z \uparrow_{q_i}^{q_1}| < \pi$ .  $\square$

### Proving $k \leq 4$ when $n = 2$ in (iv) of Theorem 2.1:

We consider  $\Sigma_{q_1} M \in \mathcal{A}^1(1)$ . By Lemma 1.1, any  $\{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_k}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_1} M$ . If  $k > 4$ , then by Theorem 0.2 we have  $k = 5$  and each  $\uparrow_{q_1}^{q_i}$  has an opposite direction  $\uparrow_{q_1}^{q_j}$  (i.e.  $|\uparrow_{q_1}^{q_i} \uparrow_{q_1}^{q_j}| = \pi$ ), where  $2 \leq i, j \leq 5$ . Due to this, we can select geodesics  $[q_1 q_2]$  and  $[q_1 q_3]$  such that  $|\uparrow_{q_1}^{q_2} \uparrow_{q_1}^{q_3}| = \pi$ ; so by Lemma 2.6, we can select geodesics  $[q_2 q_i]$  for  $i = 4$  and 5 such that  $\frac{\pi}{2} \leq |\uparrow_{q_2}^{q_i} \uparrow_{q_2}^{q_1}| < \pi$ . Then we select an arbitrary geodesic  $[q_2 q_3]$ , and consider  $B \triangleq \{\uparrow_{q_2}^{q_1}, \uparrow_{q_2}^{q_3}, \uparrow_{q_2}^{q_4}, \uparrow_{q_2}^{q_5}\}$ . Similarly, each element of  $B$  has an opposite direction in  $B$ . It then follows that  $|\uparrow_{q_2}^{q_1} \uparrow_{q_2}^{q_3}| = \pi$ , and similarly we can conclude that  $|\uparrow_{q_3}^{q_1} \uparrow_{q_3}^{q_2}| = \pi$ . Now we let  $\mathcal{S} \triangleq [q_1 q_2] \cup [q_2 q_3] \cup [q_3 q_1]$ . By Lemma 2.2,  $\mathcal{S}^{\geq \frac{\pi}{2}}$  is convex in  $M$ ; and by Lemma 2.4,  $\mathcal{S}^{\geq \frac{\pi}{2}} = \mathcal{S}^{=\frac{\pi}{2}}$ . Then by Lemma 2.5,  $\dim(\mathcal{S}^{=\frac{\pi}{2}}) = 0$ . Note that  $q_4, q_5 \in \mathcal{S}^{\geq \frac{\pi}{2}}$  because  $q_4, q_5 \in \{q_1, q_2, q_3\}^{\geq \frac{\pi}{2}}$ . Since  $\mathcal{S}^{\geq \frac{\pi}{2}}$  is convex in  $M$ , it has to hold that  $\mathcal{S}^{\geq \frac{\pi}{2}} = \{q_4, q_5\}$  with  $|q_4 q_5| = \pi$  which contradicts  $|q_4 q_5| < \pi$ . I.e., we can conclude that  $k \leq 4$ .  $\square$

### Proving $k \leq 6$ when $n = 3$ in (iv) of Theorem 2.1:

Similarly, if  $k > 6$ , by Lemma 1.1 any  $\{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_7}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_1} M$ ; and each  $\uparrow_{q_1}^{q_i}$  has an opposite direction  $\uparrow_{q_1}^{q_j}$ , where  $2 \leq i, j \leq 7$ . And we can find  $\mathcal{S} \triangleq [q_1 q_2] \cup [q_2 q_3] \cup [q_3 q_1]$  with  $|\uparrow_{q_1}^{q_2} \uparrow_{q_1}^{q_3}| = \pi$ ,  $|\uparrow_{q_2}^{q_1} \uparrow_{q_2}^{q_3}| = \pi$  and  $|\uparrow_{q_3}^{q_1} \uparrow_{q_3}^{q_2}| = \pi$ .



By Lemma 2.2,  $\mathcal{S}^{\geq \frac{\pi}{2}}$  is convex in  $M$ , which implies that  $\mathcal{S}^{\geq \frac{\pi}{2}} \in \mathcal{A}^m(1)$  for some  $m$ ; and by Lemma 2.4,  $\mathcal{S}^{\geq \frac{\pi}{2}} = \mathcal{S}^{\frac{\pi}{2}}$ . Then by Lemma 2.5, we have  $m \leq 1$ . Note that  $q_4, q_5, q_6, q_7$  belongs to  $\mathcal{S}^{\geq \frac{\pi}{2}}$  with  $\frac{\pi}{2} \leq |q_i q_j| < \pi$  for any  $4 \leq i \neq j \leq 7$ , which contradicts  $\dim(\mathcal{S}^{\geq \frac{\pi}{2}}) \leq 1$ .  $\square$

### 2.3 Estimating $k$ for $n \geq 4$ in (iv) of Theorem 2.1

When we estimate  $k$  for larger  $n$  in (iv) of Theorem 2.1, the arguments in the proofs for  $n = 2, 3$  fail. This is because some  $\uparrow_{q_1}^{q_i} \in \{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_k}\}$  maybe have no opposite direction in  $\{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_k}\}$  for larger  $n$ . In order to overcome this, we need the following proposition.

**Lemma 2.7.** *Let  $N$  be a convex subset in  $M \in \mathcal{A}^n(1)$  with  $\dim(N) = n - 1$ . If  $N^{\frac{\pi}{2}} \neq \emptyset$ , then  $N^{\frac{\pi}{2}} = \{p\}$  or  $\{p_1, p_2\}$ , and*

- (i) *if  $N^{\frac{\pi}{2}} = \{p\}$  (resp.  $\{p_1, p_2\}$ ), then there are at most two (resp. a unique) geodesics between  $p$  (resp.  $p_i$ ) and any interior point  $x$  of  $N$ ;*
- (ii) *if  $N^{\frac{\pi}{2}} = \{p_1, p_2\}$ , and if  $N$  is complete and  $N$  has empty boundary, then  $M = \{p_1, p_2\} * N$ ;*
- (iii) *if  $N^{\frac{\pi}{2}} = \{p_1, p_2\}$ , then  $\{p_1, p_2\}^{\geq \frac{\pi}{2}} = \{p_1, p_2\}^{\frac{\pi}{2}}$ .*

*Proof.* We will prove this by the induction on  $n$ . Obviously, the lemma is true if  $n = 1$ , so we assume  $n > 1$ .

Note that by the first variation formula ([BGP]), for any  $y \in N^{\frac{\pi}{2}}$ ,  $x \in N^\circ$  and any geodesic  $[yx]$ , we have

$$|\uparrow_x^y \xi| \geq \frac{\pi}{2}, \forall \xi \in \Sigma_x N.$$

Note that  $\Sigma_x N$  is convex in  $\Sigma_x M$  because  $N$  is convex in  $M$ , and  $\Sigma_x N$  has empty boundary because  $x \in N^\circ$  ([BGP]). Then by Lemma 2.3,  $|\uparrow_x^y \xi| = \frac{\pi}{2}$  in fact, i.e.,  $\uparrow_x^y \in (\Sigma_x N)^{\frac{\pi}{2}}$ ; and by the inductive assumption we know that  $(\Sigma_x N)^{\frac{\pi}{2}}$  contains at most two points in  $\Sigma_x M$ . Hence, we can conclude that  $N^{\frac{\pi}{2}} = \{p\}$  or  $\{p_1, p_2\}$ , and that (i) holds.

(ii) Due to (i), this is a special case of Proposition 2.8 below.

(iii) Given any fixed  $x \in N^\circ$ , from the above, we know that  $(\Sigma_x N)^{\frac{\pi}{2}} = \{\uparrow_x^{p_1}, \uparrow_x^{p_2}\}$ . Note that  $\Sigma_x N$  is convex and complete in  $\Sigma_x M$  and has empty boundary. Then by (ii), we have  $\Sigma_x M = \{\uparrow_x^{p_1}, \uparrow_x^{p_2}\} * \Sigma_x N$ . Hence, for any  $z \in \{p_1, p_2\}^{\geq \frac{\pi}{2}}$  and a geodesic  $[xz]$ , without loss of generality we can assume that  $|\uparrow_x^{p_1} \uparrow_x^z| \leq \frac{\pi}{2}$ . On the other hand, by applying Theorem 0.4 on  $\triangle p_1 x z$ , we have  $|\uparrow_x^{p_1} \uparrow_x^z| \geq \frac{\pi}{2}$  (note that  $|p_1 z| \geq \frac{\pi}{2}$  and  $|p_1 x| = \frac{\pi}{2}$ ). It then follows that  $|\uparrow_x^{p_1} \uparrow_x^z| = \frac{\pi}{2}$ , which implies that  $|\uparrow_x^{p_2} \uparrow_x^z| = \frac{\pi}{2}$  too. Again by applying Theorem 0.4 on  $\triangle p_i x z$  ( $i = 1, 2$ ), we conclude that  $|p_i z| = \frac{\pi}{2}$ , i.e.,  $\{p_1, p_2\}^{\geq \frac{\pi}{2}} = \{p_1, p_2\}^{\frac{\pi}{2}}$ .  $\square$

**Proposition 2.8.** *Let  $X, Y$  be two complete convex subsets in  $M \in \mathcal{A}^n(1)$ . Then*

*$M = X * Y$  if (i)  $\dim(X) + \dim(Y) + 1 = n$ ;*

*(ii)  $X$  and  $Y$  have empty boundary;*

- (iii)  $|xy| = \frac{\pi}{2}$  for any  $x \in X$  and  $y \in Y$ ;
- (iv) there is a unique geodesic between any  $x \in X$  and  $y \in Y$ .

Proposition 2.8 is due to the definition of the metric of the join ([BGP]). For a detailed proof, one can refer to [SSW].

Now we prove that, in (iv) of Theorem 2.1,  $k$  has the desired upper bound for  $n \geq 4$ .

**Proof for the estimate of  $k$  for  $n \geq 4$  in (iv) of Theorem 2.1:**

We consider  $\Sigma_{q_1} M \in \mathcal{A}^{n-1}(1)$ . By Lemma 1.1, any  $\{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_k}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_1} M$ . If  $\{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_k}\}$  has no two opposite directions (i.e.  $|\uparrow_{q_1}^{q_i} \uparrow_{q_1}^{q_j}| < \pi$  for any  $2 \leq i \neq j \leq k$ ), then by induction we have that  $k - 1 \leq 3(l - 1) + 1$  (resp.  $3l$ ) for  $n - 1 = 2l - 2$  (resp.  $2l - 1$ ), i.e.,  $k \leq 3l - 1$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ). Hence,  $k \leq 3l$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ).

From the above, we conclude that if  $k > 3l$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ) (Hint: In fact, “ $k > 3l$ ” can be changed to “ $k \geq 3l$ ” when  $n = 2l - 1$ ), then

$$\text{for any } 1 \leq i \leq k, \text{ any } \{\uparrow_{q_i}^{q_j} \mid 1 \leq j \leq k, j \neq i\} \text{ has two opposite directions.} \quad (2.9)$$

Now we assume that  $k > 3l$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ), and we consider  $\{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_k}\}$ . By (2.9), without loss of generality, we can assume that

$$|\uparrow_{q_1}^{q_2} \uparrow_{q_1}^{q_3}| = \pi.$$

By Lemma 2.2,  $X \triangleq \{q_1, q_2, q_3\}^{\geq \frac{\pi}{2}}$  is convex in  $M$ , so  $X \in \mathcal{A}^m(1)$  for some  $m$ . And by Lemma 2.6,  $X \subseteq \{q_2, q_3\}^{\frac{\pi}{2}}$ , so  $m \leq n - 1$  by Lemma 2.5. Note that  $\{q_4, \dots, q_k\}$  is a  $\frac{\pi}{2}$ -separated subset in  $X$ .

If  $m \leq n - 2$ , then  $m \leq 2l - 3$  (resp.  $2l - 2$ ) for  $n = 2l - 1$  (resp.  $2l$ ). By the inductive assumption on  $X$ , we conclude that  $k - 3 \leq 3(l - 1)$  (resp.  $3(l - 1) + 1$ ) for  $m \leq 2l - 3$  (resp.  $2l - 2$ ), which contradicts the assumption “ $k > 3l$  (resp.  $3l + 1$ )”.

If  $m = n - 1$ , then by Lemma 2.7 it has to hold that  $X^{\frac{\pi}{2}} = \{q_2, q_3\}$ . And by (iii) and (i) of Lemma 2.7, we have  $Y \triangleq \{q_2, q_3\}^{\geq \frac{\pi}{2}} = \{q_2, q_3\}^{\frac{\pi}{2}}$ , and there is a unique geodesic between  $q_2$  and any interior point of  $Y$ . This implies that  $\{q_2\} * Y$  can be isometrically embedded into  $M$  (note that  $Y$  is convex in  $M$  by Lemma 2.2, and see Proposition 2.8 or refer to [SSW]). Note that  $\{q_1, q_4, \dots, q_k\} \subset Y$ . Then we can find a  $\frac{\pi}{2}$ -separated subset  $\{\uparrow_{q_2}^{q_1}, \uparrow_{q_2}^{q_4}, \dots, \uparrow_{q_2}^{q_k}\}$  in  $\Sigma_{q_2} M$  with  $|\uparrow_{q_2}^{q_i} \uparrow_{q_2}^{q_j}| = |q_i q_j|$  for any  $i, j \in \{1, 4, \dots, k\}$ . Since  $|q_i q_j| < \pi$ , due to (2.9) there is a geodesic  $[q_2 q_3]$  such that  $|\uparrow_{q_2}^{q_{i_0}} \uparrow_{q_2}^{q_3}| = \pi$  for some  $i_0 \in \{1, 4, \dots, k\}$ . By Lemma 2.6,  $Z \triangleq Y \cap \{q_{i_0}\}^{\geq \frac{\pi}{2}}$  belongs to  $Y \cap \{q_{i_0}\}^{\frac{\pi}{2}}$ , which implies that  $\dim(Z) \leq m - 1 = n - 2$  by Lemma 2.5 (note that  $Z$  is convex in  $Y$  (and  $M$ )). By the inductive assumption on  $Z$  (note that  $\{q_1, \dots, q_k\} \setminus \{q_2, q_3, q_{i_0}\}$  is a  $\frac{\pi}{2}$ -separated subset of  $Z$ ), we conclude that  $k - 3 \leq 3(l - 1)$  (resp.  $3(l - 1) + 1$ ) for  $n - 2 = 2l - 3$  (resp.  $2l - 2$ ), which contradicts the assumption “ $k > 3l$  (resp.  $3l + 1$ )”.

Since all contradictions are gotten under the assumption “ $k > 3l$  (resp.  $3l + 1$ )” for  $n = 2l - 1$  (resp.  $2l$ ), we conclude that  $k \leq 3l$  (resp.  $3l + 1$ ).  $\square$

**Remark 2.10.** In proving that  $k$  has the desired upper bound in (iii) of Theorem 2.1, a main difference to the above proof is that we will consider  $\{\uparrow_{q_{h+1}}^{q_1}, \dots, \uparrow_{q_{h+1}}^{q_h}, \uparrow_{q_{h+1}}^{q_{h+2}}, \dots, \uparrow_{q_{h+1}}^{q_k}\}$  in  $\Sigma_{q_{h+1}}M$  (here it will be better to point out that, by Lemma 1.2, it is not hard to see that we have  $i, j > h$  if  $|\uparrow_{q_{h+1}}^{q_i} \uparrow_{q_{h+1}}^{q_j}| = \pi$ ).

## 2.4 Proof for the structure classification in (iv) of Theorem 2.1

**Proof for  $n = 2$  and  $k = 4$ :**

We only need to prove that there is a point in  $\{q_1, \dots, q_4\}$ , say  $q_4$ , such that  $M = \{q_4\} * S^1$  with  $\{q_1, q_2, q_3\} \subset S^1$  if  $M$  has nonempty boundary.

**Claim 1:** *There is a  $q_i$  with  $q_i \in \partial M$ .* If  $|q_1 \partial M| \geq \frac{\pi}{2}$ , then it is easy to see that, by Lemma 1.4,  $q_i \in \partial M$  for  $2 \leq i \leq 4$ . If  $|q_1 \partial M| < \frac{\pi}{2}$ , we select  $p \in \partial M$  such that  $|q_1 p| = |q_1 \partial M|$ . Like Proof 1.3, we can prove that  $\Sigma_p M = \{\uparrow_p^{q_1}\} * \partial(\Sigma_p M)$ , and that  $|pq_j| \geq \frac{\pi}{2}$  for  $j \neq 1$ . If  $|pq_j| = \pi$  for some  $j \neq 1$ , then  $M = \{p, q_j\} * A$  for some arc  $A$  (Lemma 1.2) which implies  $q_j \in \partial M$ . If  $|pq_j| < \pi$  for  $j = 2, 3$  and  $4$ , then by Lemma 1.1, any  $\{\uparrow_p^{q_2}, \uparrow_p^{q_3}, \uparrow_p^{q_4}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_p M (= \{\uparrow_p^{q_1}\} * \partial(\Sigma_p M)) \in \mathcal{A}^1(1)$ . It therefore follows that, without loss of generality, we can assume that  $\uparrow_p^{q_2}$  and  $\uparrow_p^{q_3}$  belong to  $\partial(\Sigma_p M)$  with  $|\uparrow_p^{q_2} \uparrow_p^{q_3}| = \pi$ . Note that  $\uparrow_p^{q_2} \in \partial(\Sigma_p M)$  implies that there exists a geodesic  $[pq_2]$  belonging to  $\partial M$  ([BGP]), i.e., Claim 1 is verified.

**Claim 2:** *There is a  $q_i$  with  $q_i \notin \partial M$ .* If this is not true, we select any  $q_i$ , say  $q_2$ , and consider  $\Sigma_{q_2} M \in \mathcal{A}^1(1)$  which has nonempty boundary. By Lemma 1.1, any  $\{\uparrow_{q_2}^{q_1}, \uparrow_{q_2}^{q_3}, \uparrow_{q_2}^{q_4}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_2} M$ ; so without loss of generality we can assume that  $\uparrow_{q_2}^{q_1}$  and  $\uparrow_{q_2}^{q_3}$  belong to  $\partial(\Sigma_{q_2} M)$  with  $|\uparrow_{q_2}^{q_1} \uparrow_{q_2}^{q_3}| = \pi$ . This implies that there exists a unique geodesic between  $q_2$  and  $q_k$  ( $k = 1, 3$ ) and  $[q_2 q_k]$  belongs to  $\partial M$  (Hint: if  $|pq| = \pi$  for  $p, q \in M \in \mathcal{A}^n(1)$ , then  $q$  is the unique point in  $M$  such that  $|pq| = \pi$  (Lemma 1.2)). Hence, if Claim 2 is not true, then we can rearrange  $q_1, \dots, q_4$  such that there is a unique geodesic  $[q_i q_{i+1}]$  between  $q_i$  and  $q_{i+1}$  for  $i = 1, \dots, 4$  (where  $q_5 = q_1$ ) with  $\cup_{i=1}^4 [q_i q_{i+1}] \subseteq \partial M$  and  $|\uparrow_{q_i}^{q_{i-1}} \uparrow_{q_i}^{q_{i+1}}| = \pi$  (where  $q_0 = q_4$ ). By Lemma 2.6 and its proof, we have  $|\uparrow_{q_1}^{q_3} \uparrow_{q_1}^{q_2}| = |\uparrow_{q_1}^{q_3} \uparrow_{q_1}^{q_4}| = |\uparrow_{q_3}^{q_1} \uparrow_{q_3}^{q_2}| = \frac{\pi}{2}$  and  $|q_3 q_2| = \frac{\pi}{2}$ . Then by Theorem 0.5, there is a geodesic  $[q_1 q_3]$  such that the triangle  $\triangle q_2 q_1 q_3$  (formed by  $[q_1 q_2]$ ,  $[q_2 q_3]$  and  $[q_1 q_3]$ ) is isometric to its comparison triangle, so  $|\uparrow_{q_2}^{q_1} \uparrow_{q_2}^{q_3}| = |q_1 q_3| < \pi$  which contradicts  $|\uparrow_{q_2}^{q_1} \uparrow_{q_2}^{q_3}| = \pi$ . Hence, Claim 2 has to hold.

Due to Claims 1 and 2, we can assume that  $q_4 \notin \partial M$  and  $q_2 \in \partial M$ . Then by the proof of Claim 2, we can conclude that  $q_1, q_3 \in \partial M$  too, and there are geodesics  $[q_1 q_2]$ ,  $[q_2 q_3]$  and  $[q_3 q_1]$  such that  $\mathcal{S} \triangleq [q_1 q_2] \cup [q_2 q_3] \cup [q_3 q_1] \subseteq \partial M$  and  $|\uparrow_{q_1}^{q_3} \uparrow_{q_1}^{q_2}| = |\uparrow_{q_2}^{q_1} \uparrow_{q_2}^{q_3}| = |\uparrow_{q_3}^{q_2} \uparrow_{q_3}^{q_1}| = \pi$ . By Lemma 2.4, we have  $\mathcal{S}^{\geq \frac{\pi}{2}} = \mathcal{S}^{\frac{\pi}{2}}$ . Since  $|q_4 q_i| \geq \frac{\pi}{2}$  for  $i = 1, 2, 3$ , by Theorem 0.4 we have  $q_4 \in \mathcal{S}^{\geq \frac{\pi}{2}}$ , and so  $|q_4 p| = \frac{\pi}{2}$  for all  $p \in \mathcal{S}$ . By the first variation formula ([BGP]) together with  $\mathcal{S} \subseteq \partial M$ , we have that any geodesic  $[q_4 p]$  is perpendicular to  $\mathcal{S}$  at any  $p \in \mathcal{S}$ , and that there is a unique direction which is perpendicular to  $\mathcal{S}$  at any  $p \in \mathcal{S}$ . This implies that there is a unique geodesic between  $q_4$  and any  $p \in \mathcal{S}$ , and that any point in  $M$  lies in a geodesic  $[q_4 p]$  for some  $p \in \mathcal{S}$ .

Then by Theorem 0.5, we can conclude that  $M = \{q_4\} * \mathcal{S}$ , i.e.,  $M = \{q_4\} * S^1$  with  $\{q_1, q_2, q_3\} \subset S^1$ .  $\square$

In the proof for  $n \geq 3$ , we need a more technical result than Proposition 2.8.

**Lemma 2.11.** *Let  $X$  and a circle  $S^1$  be two convex subsets in  $M \in \mathcal{A}^n(1)$ . Then  $M = X * S^1$  if (i)  $\dim(X) = n - 2$ ; (ii)  $X$  is complete and has empty boundary; (iii) the perimeter of  $S^1$  is bigger than  $\pi$ ; (iv)  $|xy| = \frac{\pi}{2}$  for any  $x \in X$  and  $y \in S^1$ .*

In Lemma 2.11, if condition (iii) is canceled, Rong-Wang proved that there are  $\hat{X} \in \mathcal{A}^{n-2}(1)$ ,  $\hat{S}^1 \in \mathcal{A}^1(1)$  and a cyclic group  $\Gamma$  which acts by isometries on  $\hat{X}$  and  $\hat{S}^1$  such that  $X = \hat{X}/\Gamma$ ,  $S^1 = \hat{S}^1/\Gamma$  and  $M = (\hat{X} * \hat{S}^1)/\Gamma$  (cf. [RW]). Note that this implies Lemma 2.11. However, for the convenience of readers, we will give the proof of Lemma 2.11 in Appendix.

Now we prove the latter part of (iv) for  $n \geq 3$  in Theorem 2.1.

**Proof for  $n \geq 3$ :**

Case 1:  $n = 2l - 1$  and  $k = 3l$  with  $l \geq 2$ .

Note that (2.9) still holds when  $n = 2l - 1$  and  $k = 3l$  (see the hint before (2.9)). Like the proof for the estimate of  $k$  for  $n \geq 4$  in (iv) of Theorem 2.1, we can select  $q_1, q_2, q_3$  such that  $X \triangleq \{q_1, q_2, q_3\}^{\geq \frac{\pi}{2}} \in \mathcal{A}^m(1)$  belongs to  $\{q_2, q_3\}^{\frac{\pi}{2}}$ . Since  $X$  is convex in  $M$  (Lemma 2.2), we have that  $m \leq n - 1$  (Lemma 2.5). Note that  $\{q_4, \dots, q_k\}$  is a  $\frac{\pi}{2}$ -separated subset in  $X$  with  $|q_i q_j| < \pi$  ( $i \neq j$ ), which implies that  $m \geq n - 2$  (by the estimate of  $k$  in (iv) of Theorem 2.1). It then follows that  $m = n - 2$  or  $n - 1$ .

If  $m = n - 2$ , then by induction we conclude that  $X$  is isometric to  $S_2^1 * \dots * S_l^1$ , where  $S_j^1$  ( $2 \leq j \leq l$ ) has perimeter  $\geq \frac{3\pi}{2}$  (and we can rearrange  $q_4, \dots, q_{3l}$  such that  $q_{3j-2}, q_{3j-1}, q_{3j} \in S_j^1$ ). Of course,  $X$  has empty boundary. Since  $X$  is convex in  $M$ , by Lemmas 2.3 and 2.2,  $X^{\geq \frac{\pi}{2}} = X^{\frac{\pi}{2}}$  and  $X^{\geq \frac{\pi}{2}}$  is convex in  $M$ , and so by Lemma 2.5  $\dim(X^{\frac{\pi}{2}}) \leq 1$ . Obviously,  $\{q_1, q_2, q_3\} \subset X^{\frac{\pi}{2}}$ , so it has to hold that  $X^{\frac{\pi}{2}}$  is a circle with perimeter  $\geq \frac{3\pi}{2}$ , denoted by  $S_1^1$ . Then by Lemma 2.11, it follows that  $M$  is isometric to  $S_1^1 * \dots * S_l^1$  with  $q_{3j-2}, q_{3j-1}, q_{3j} \in S_j^1$  for  $1 \leq j \leq l$ .

If  $m = n - 1$ , like the case “ $m = n - 1$ ” in the proof for the estimate of  $k$  for  $n \geq 4$  in (iv) of Theorem 2.1, we have that  $X^{\frac{\pi}{2}} = \{q_2, q_3\}$  and  $Y \triangleq \{q_2, q_3\}^{\geq \frac{\pi}{2}} = \{q_2, q_3\}^{\frac{\pi}{2}}$ . And we can select  $q_{i_0}$  ( $i_0 \neq 2, 3$ ) such that  $Z \triangleq Y \cap \{q_{i_0}\}^{\geq \frac{\pi}{2}} = Y \cap \{q_{i_0}\}^{\frac{\pi}{2}}$ ; and  $Z$  is convex in  $M$  with  $\dim(Z) \leq m - 1 = n - 2$ ; and  $\{q_1, \dots, q_{3l}\} \setminus \{q_2, q_3, q_{i_0}\}$  is a  $\frac{\pi}{2}$ -separated subset of  $Z$ . By induction,  $Z$  is isometric to  $S_2^1 * \dots * S_l^1$ , and similarly we can derive that  $M$  is isometric to  $S_1^1 * \dots * S_l^1$  with  $S_j^1$  having perimeter  $\geq \frac{3\pi}{2}$  (and we can rearrange  $q_1, \dots, q_{3l}$  such that  $q_{3j-2}, q_{3j-1}, q_{3j} \in S_j^1$  for  $1 \leq j \leq l$ ).

Case 2:  $n = 2l$  and  $k = 3l + 1$  with  $l \geq 2$ .

In this case, (2.9) may not hold. We will give discussions according to “(2.9) holds” and “(2.9) does not hold”.

Subcase 1: (2.9) holds.

Like in Case 1 ( $n = 2l - 1$  and  $k = 3l$ ), we can find convex and complete subset  $X$  or  $Z \in \mathcal{A}^{n-2}(1)$  in which  $\{q_4, \dots, q_{3l+1}\}$  or  $\{q_1, \dots, q_{3l+1}\} \setminus \{q_{i_0}, q_2, q_3\}$  are  $\frac{\pi}{2}$ -separated subsets respectively. Without loss of generality, we assume that such an  $X$  is found.

If  $X$  has empty boundary, then  $X^{\geq \frac{\pi}{2}} = X = \frac{\pi}{2}$  (Lemma 2.3) which is convex in  $M$  (Lemma 2.2), and thus  $\dim(X^{\geq \frac{\pi}{2}}) \leq 1$  (Lemma 2.5). Note that  $\{q_1, q_2, q_3\}$  is a  $\frac{\pi}{2}$ -separated subset of  $X^{\geq \frac{\pi}{2}}$ , so it has to hold that  $X^{\geq \frac{\pi}{2}}$  is a circle  $S^1$  with perimeter  $\geq \frac{3\pi}{2}$ . Hence, by Lemma 2.11 we have  $M = X * S^1$ . This implies that  $M$  has the desired structure because  $X \in \mathcal{A}^{n-2}(1)$  has the desired structure by induction.

If  $X$  has nonempty boundary, then by induction we can rearrange  $q_4, \dots, q_{3l+1}$  such that  $X$  is isometric to  $\{q_{3l+1}\} * S_2^1 * \dots * S_l^1$  with  $S_j^1$  having perimeter  $\geq \frac{3\pi}{2}$  and  $q_{3j-2}, q_{3j-1}, q_{3j} \in S_j^1$  for  $2 \leq j \leq l$ . Now we consider  $(S_2^1)^{\geq \frac{\pi}{2}}$ . Note that  $S_2^1$  is convex in  $M$  (because  $X (= \{q_{3l+1}\} * S_2^1 * \dots * S_l^1)$  is convex in  $M$ ). Then  $W \triangleq (S_2^1)^{\geq \frac{\pi}{2}}$  is convex in  $M$  (Lemma 2.2), and  $W = (S_2^1)^{= \frac{\pi}{2}}$  (Lemma 2.3), and  $\dim(W) \leq n - 2$  (Lemma 2.5). Note that  $\{q_1, q_2, q_3, q_7, \dots, q_{3l+1}\}$  is a  $\frac{\pi}{2}$ -separated subset of  $W$ , so  $\dim(W) = n - 2$  and  $W \in \mathcal{A}^{n-2}(1)$  has the desired structure by induction. If  $W$  has empty boundary, then by Lemma 2.11  $M$  is isometric to  $S_2^1 * W$ , and thus  $M$  has the desired structure. If  $W$  has nonempty boundary, then by induction we can assume that  $W = \{q_{3l+1}\} * S_1^1 * S_3^1 * \dots * S_l^1$  with  $q_1, q_2, q_3 \in S_1^1$ , and from the proof of Lemma 2.11 (see Remark A.1 in Appendix)  $S_2^1 * W$  can be isometrically embedded into  $M$ . And it is not hard to see that  $M = \{q_{3l+1}\} * S_1^1 * S_2^1 * \dots * S_l^1$  if  $M$  has nonempty boundary.

Subcase 2: (2.9) does not hold.

In the proof, we will need the following lemma.

**Lemma 2.12.** *Let  $X * Y$  be a join with  $X, Y \in \mathcal{A}(1)$ . If  $Q \triangleq \{q_1, \dots, q_k\}$  is a  $\frac{\pi}{2}$ -net of  $X$  (i.e. for any  $x \in X$  there is  $q_i \in Q$  such that  $|xq_i| < \frac{\pi}{2}$ ), then  $Q^{\geq \frac{\pi}{2}} = Y$ .*

Proof. It sufficed to show that  $Q^{\geq \frac{\pi}{2}} \subset Y$ . Let  $p$  be any point in  $Q^{\geq \frac{\pi}{2}}$ . If  $p \notin Y$ , then  $p = [(x, y, t)]$  with  $x \in X$ ,  $y \in Y$  and  $t < \frac{\pi}{2}$ . Select  $q_i \in Q$  such that  $|xq_i| < \frac{\pi}{2}$  and a geodesic  $[xq_i]$ . By the definition of the metric of the join, we know that  $|\uparrow_x^{q_i} \uparrow_x^p| = \frac{\pi}{2}$  (cf. [SSW]). Then by Theorem 0.4 on the triangle  $\triangle pxq_i$ , we conclude that  $|pq_i| < \frac{\pi}{2}$  (note that  $|xp| = t < \frac{\pi}{2}$  and  $|xq_i| < \frac{\pi}{2}$ ) which contradicts  $p \in Q^{\geq \frac{\pi}{2}}$ . Hence, it has to hold that  $p \in Y$ . ■

Now we continue the proof under the assumption “(2.9) does not hold”.

Since (2.9) does not hold, without loss of generality, we can select  $\{\uparrow_{q_{3l+1}}^{q_1}, \dots, \uparrow_{q_{3l+1}}^{q_{3l}}\}$  in  $\Sigma_{q_{3l+1}} M \in \mathcal{A}^{n-1}(1)$  such that  $|\uparrow_{q_{3l+1}}^{q_i} \uparrow_{q_{3l+1}}^{q_j}| < \pi$  for any  $1 \leq i \neq j \leq 3l$ . On the other hand,  $\{\uparrow_{q_{3l+1}}^{q_1}, \dots, \uparrow_{q_{3l+1}}^{q_{3l}}\}$  is a  $\frac{\pi}{2}$ -separated subset of  $\Sigma_{q_{3l+1}} M \in \mathcal{A}^{n-1}(1)$  (Lemma 1.1). Then by the case “ $n = 2l - 1$  and  $k = 3l$ ” (here  $n - 1 = 2l - 1$ ), we can conclude that  $\Sigma_{q_{3l+1}} M = \bar{S}_1^1 * \dots * \bar{S}_l^1$  with  $\bar{S}_i^1$  having perimeter  $\geq \frac{3\pi}{2}$  and  $\uparrow_{q_{3l+1}}^{q_{3i-2}}, \uparrow_{q_{3l+1}}^{q_{3i-1}}, \uparrow_{q_{3l+1}}^{q_{3i}} \in \bar{S}_i^1$ .

Let  $A \triangleq \{q_{3l+1}, q_{3l}, q_{3l-1}, q_{3l-2}\}^{\geq \frac{\pi}{2}}$ , which is convex in  $M$  (Lemma 2.2). If there is  $p \in A$  such that  $|q_{3l+1}p| = \pi$ , then  $M = \{q_{3l+1}, p\} * L$  for some  $L \in \mathcal{A}^{n-1}(1)$  (Lemma 1.2). Note that we can assume that  $\{q_1, \dots, q_{3l}\}$  is a  $\frac{\pi}{2}$ -separated subset of  $L$  (otherwise we can replace  $q_i$  with the the point  $[pq_i] \cap L$  for  $i = 1, 2, \dots, 3l$ ). Similarly,

by the case “ $n = 2l - 1$  and  $k = 3l$ ”, we can conclude that  $L = S_1^1 * \cdots * S_l^1$  with  $S_i^1$  having perimeter  $\geq \frac{3\pi}{2}$  and  $q_{3i-2}, q_{3i-1}, q_{3i} \in S_i^1$ , and so  $\{q_{3l+1}\} * S_1^1 * \cdots * S_l^1$  can be isometrically embedded into  $M$ . Now we assume that  $|q_{3l+1}p| < \pi$  for any  $p \in A$ . By Lemma 1.1, for any geodesic  $[q_{3l+1}p]$ , we have  $|\uparrow_{q_{3l+1}}^p \uparrow_{q_{3l+1}}^{q_j}| \geq \frac{\pi}{2}$  for  $j = 3l-2, 3l-1, 3l$ . Note that  $\{\uparrow_{q_{3l+1}}^{q_{3l-2}}, \uparrow_{q_{3l+1}}^{q_{3l-1}}, \uparrow_{q_{3l+1}}^{q_{3l}}\}$  is a  $\frac{\pi}{2}$ -net of  $\bar{S}_l^1$ , so by Lemma 2.12

$$\uparrow_{q_{3l+1}}^p \in \bar{S}_1^1 * \cdots * \bar{S}_{l-1}^1.$$

Next we consider  $B \triangleq (A \cup \{q_{3l+1}\})^{\geq \frac{\pi}{2}}$ , which is also convex in  $M$ . Similarly, we assume that  $|q_{3l+1}r| < \pi$  for any  $r \in B$ ; so for any geodesic  $[q_{3l+1}r]$ ,  $|\uparrow_{q_{3l+1}}^r \uparrow_{q_{3l+1}}^{q_j}| \geq \frac{\pi}{2}$  for any  $1 \leq j \leq 3l-3$ . And it is not hard to see that  $\{\uparrow_{q_{3l+1}}^{q_1}, \dots, \uparrow_{q_{3l+1}}^{q_{3l-3}}\}$  is a  $\frac{\pi}{2}$ -net of  $\bar{S}_1^1 * \cdots * \bar{S}_{l-1}^1$ , so by Lemma 2.12

$$\uparrow_{q_{3l+1}}^r \in \bar{S}_l^1.$$

It therefore follows that  $|\uparrow_{q_{3l+1}}^p \uparrow_{q_{3l+1}}^r| = \frac{\pi}{2}$ . Then by Theorem 0.4 on any  $\triangle q_{3l+1}pr$ , we have  $|pr| = \frac{\pi}{2}$  (note that  $|q_{3l+1}p|, |q_{3l+1}r|, |pr| \geq \frac{\pi}{2}$ ). Hence, by Lemma 2.5, we have

$$\dim(A) + \dim(B) \leq n - 1.$$

Note that  $\{q_1, \dots, q_{3l-3}\} \subset A$  and  $\{q_{3l-2}, q_{3l-1}, q_{3l}\} \subset B$ , and  $A$  and  $B$  are convex in  $M$ . This implies that  $\dim(A) \geq n - 3$  (see Subsection 2.3) and  $\dim(B) \geq 1$ . It then has to hold that either  $\dim(A) = n - 3$ , or  $\dim(A) = n - 2$  and  $\dim(B) = 1$ .

If  $\dim(A) = n - 3$ , then by the case “ $n = 2l - 1$  and  $k = 3l$ ” (here  $n - 3 = 2l - 3$ ) we get that  $A = S_1^1 * \cdots * S_{l-1}^1$  with  $S_i^1$  having perimeter  $\geq \frac{3\pi}{2}$  and  $q_{3i-2}, q_{3i-1}, q_{3i} \in S_i^1$ . If  $\dim(A) = n - 2$  and  $\dim(B) = 1$ , then  $B (\ni q_{3l-2}, q_{3l-1}, q_{3l})$  is a circle with perimeter  $\geq \frac{3\pi}{2}$ . Now we let  $\mathcal{S}$  denote  $S_1^1$  or the circle  $B$  (which is convex in  $M$ ), and we consider  $C \triangleq (\mathcal{S})^{\geq \frac{\pi}{2}}$ , which is convex in  $M$  (Lemma 2.2). By Lemmas 2.3 and 2.5,  $C = (\mathcal{S})^{\frac{\pi}{2}}$  and  $\dim(C) \leq n - 2$ . Note that  $\{q_4, \dots, q_{3l+1}\} \subset C$  or  $\{q_1, \dots, q_{3l-3}, q_{3l+1}\} \subset C$ , so we have  $\dim(C) \geq n - 2$  (see Subsection 2.3). Hence, we have  $\dim(C) = n - 2$ , and so by Lemma 2.11 or its proof (see Remark A.1) we get that  $M = \mathcal{S} * C$  (if  $C$  has empty boundary) or  $\mathcal{S} * C$  can be isometrically embedded into  $M$  (if  $C$  has nonempty boundary). On the other hand,  $C$  has the desired structure by induction. Hence, it is not hard to see that  $M$  has the desired structure.  $\square$

So far we have finished the proof of Theorem 2.1 (which implies Theorem B). In the rest of this section, we will give the proof of Corollary C.

### Proof of Corollary C:

Since  $M \in \mathcal{A}^n(1)$ , by Theorem B, we have  $k \leq 3l$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ); moreover, since  $M$  has empty boundary (because  $M$  is closed) and  $n > 2$ ,  $M$  is isometric to  $S_1^1 * \cdots * S_l^1$  (resp.  $S_1^1 * \cdots * S_{l-1}^1 * N$  for some  $N \in \mathcal{A}^2(1)$  or  $\{q_{3l+1}\} * S_1^1 * \cdots * S_l^1$  can be isometrically embedded into  $M$ ) with  $l \geq 2$  if  $k = 3l$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ). Given a join  $X * Y$  with  $X, Y \in \mathcal{A}(1)$ , from the definition of the metric of the join (cf. [SSW]), we know that  $\Sigma_x(X * Y) = (\Sigma_x X) * Y$

for any  $x \in X \subset X * Y$ . Since  $M$  is a Riemannian manifold,  $\Sigma_p M$  is isometric to the unit sphere  $\mathbb{S}^{n-1}$  for any  $p \in M$ . Therefore, if  $k = 3l$  (resp.  $3l + 1$ ) for  $n = 2l - 1$  (resp.  $2l$ ), then each  $S_i^1$  is a great circle (i.e. having perimeter equal to  $2\pi$ ) which can be isometrically embedded into  $M$ . It then follows from the Maximum Diameter Theorem ([CE]) that  $M$  is isometric to the unit sphere  $\mathbb{S}^n$ .  $\square$

### 3 Proofs of Theorem D and Corollary E

We first give an interesting and key lemma, which may be known to experts.

**Lemma 3.1.** *Let  $M \in \mathcal{A}^n(1)$ , and let  $\mathbb{S}^k$  be the  $k$ -dimensional unit sphere. If there exists a noncontractive map  $f : \mathbb{S}^k \rightarrow M$ , then  $f$  is an isometrical embedding.*

*Proof.* It suffices to show that  $|f(x)f(y)| = |xy|$  for any  $x, y \in \mathbb{S}^k$ . Note that  $\text{diam}(M) \leq \pi$  because  $M \in \mathcal{A}^n(1)$  ([BGP]); so if  $|xy| = \pi$ , then  $|f(x)f(y)| = \pi = |xy|$  because  $f$  is a noncontractive map (note that  $\mathbb{S}^0$  consists of two points with distance equal to  $\pi$ ). Now we assume that  $|xy| < \pi$ . Let  $z$  be the antipodal point of the middle point of  $[xy]$  (in  $\mathbb{S}^k$ ,  $[xy]$  is the unique geodesic between  $x$  and  $y$  if  $|xy| < \pi$ ). Note that

$$|xy| + |xz| + |zy| = 2\pi. \quad (3.2)$$

Since  $f$  is a noncontractive map, we have

$$|f(x)f(y)| \geq |xy|, \quad |f(x)f(z)| \geq |xz|, \quad |f(z)f(y)| \geq |zy|, \quad (3.3)$$

and thus

$$|f(x)f(y)| + |f(x)f(z)| + |f(z)f(y)| \geq 2\pi.$$

On the other hand, because  $M \in \mathcal{A}^n(1)$ , we have ([BGP])

$$|f(x)f(y)| + |f(x)f(z)| + |f(z)f(y)| \leq 2\pi.$$

It then follows that

$$|f(x)f(y)| + |f(x)f(z)| + |f(z)f(y)| = 2\pi. \quad (3.4)$$

From (3.2)-(3.4), we derive that  $|f(x)f(y)| = |xy|$ .  $\square$

*Proof of the former part of Theorem D.*

We will give the proof by the induction on  $k$ . We first note that  $|q_i q_j| = \frac{\pi}{2}$  for any  $i \neq j$  because  $\{q_1, \dots, q_k\}$  is a  $\frac{\pi}{2}$ -separated subset and  $\text{diam}(M) \leq \frac{\pi}{2}$ . Hence, Theorem D is obviously true when  $k = 1$  and  $2$ . Now we assume that  $k \geq 3$ .

By induction, there exists an isometrical embedding  $g : \Delta_+^{k-2} \rightarrow M$  such that  $q_1, \dots, q_{k-1}$  are the vertices of  $g(\Delta_+^{k-2})$ . We denote  $g(\Delta_+^{k-2})$  by  $\Delta$ , and denote by  $\Delta^\circ$  the interior part of  $\Delta$ . By Theorem 0.4,  $\Delta$  belongs to  $\{q_k\}^{\geq \frac{\pi}{2}}$  (note that  $q_1, \dots, q_{k-1}$  are the vertices of  $\Delta$ ), which is convex in  $M$  (Lemma 2.2). On the other hand,  $\{q_k\}^{\geq \frac{\pi}{2}} =$

$\{q_k\} = \frac{\pi}{2}$  because  $\text{diam}(M) \leq \frac{\pi}{2}$ . Then by Theorem 0.5, there is a triangle  $\triangle q_k xy$  which is isometric to its comparison triangle (in  $\mathbb{S}^2$ ) for any  $x, y \in \Delta$ . (In fact, the triangle  $\triangle q_k xy$  bounds a convex domain which is isometric to the convex domain bounded by the comparison triangle of  $\triangle q_k xy$  in  $\mathbb{S}^2$  ([GM]); so if  $k = 3$ , then the proof is done.)

Now we fix a point  $p \in \Delta^\circ$  and a geodesic  $[q_k p]$ . Note that for any  $x \in \Delta$ , there is a unique geodesic  $[px]$  between  $p$  and  $x$  in  $M$  and  $[px] \subset \Delta$  because  $g$  is an isometrical embedding. From the above we know that there is a geodesic  $[q_k x]$  such that the triangle  $\triangle q_k px$  composed by  $[q_k p]$ ,  $[q_k x]$  and  $[px]$  is isometric to its comparison triangle. Then we can define a map

$$h : \Delta \rightarrow \Sigma_{q_k} M \text{ by } x \mapsto \uparrow_{q_k}^x.$$

Note that for any  $x \in \Delta$  with  $x \neq p$ , there is a unique  $y \in \partial\Delta$  such that  $x \in [py] \subset \Delta$ . Since  $\triangle q_k py$  bounds a convex domain which is isometric to the convex domain bounded by the comparison triangle of  $\triangle q_k py$  in  $\mathbb{S}^2$ , we can select  $[q_k x]$  (for all  $x \in \Delta$ ) such that  $h([px])$  is a geodesic  $[\uparrow_{q_k}^p \uparrow_{q_k}^x]$  in  $\Sigma_{q_k} M$ . Hence,  $h$  naturally induces a ‘tangential’ map

$$Dh : \Sigma_p \Delta \rightarrow \Sigma_{\uparrow_{q_k}^p}(\Sigma_{q_k} M) \text{ defined by } \uparrow_p^x \mapsto \uparrow_{\uparrow_{q_k}^p}^{\uparrow_{q_k}^x}.$$

Note that  $\Sigma_p \Delta = \mathbb{S}^{k-3}$  and  $\Sigma_{\uparrow_{q_k}^p}(\Sigma_{q_k} M) \in \mathcal{A}^{n-2}(1)$ .

**Claim:**  $Dh$  is a noncontractive map. Since triangles  $\triangle q_k px$  and  $\triangle q_k py$  are isometric to their comparison triangles respectively for any  $x, y \in \Delta$ , we have

$$|\uparrow_{q_k}^p \uparrow_{q_k}^x| = |px| \text{ and } |\uparrow_{q_k}^p \uparrow_{q_k}^y| = |py|. \quad (3.5)$$

On the other hand, by Theorem 0.4 (on the triangle  $\triangle q_k xy$ ) we have

$$|\uparrow_{q_k}^x \uparrow_{q_k}^y| \geq |xy| \quad (3.6)$$

(note that  $|q_k x| = |q_k y| = \frac{\pi}{2}$ ). Then by the definition of angles ([BGP]), we have

$$|\uparrow_{\uparrow_{q_k}^p}^{\uparrow_{q_k}^x} \uparrow_{\uparrow_{q_k}^p}^{\uparrow_{q_k}^y}| \geq |\uparrow_p^x \uparrow_p^y|, \quad (3.7)$$

i.e. the claim is verified.

By Lemma 3.1, the claim implies that  $Dh$  is an isometrical embedding in fact, so the inequality (3.7) is an equality:

$$|\uparrow_{\uparrow_{q_k}^p}^{\uparrow_{q_k}^x} \uparrow_{\uparrow_{q_k}^p}^{\uparrow_{q_k}^y}| = |\uparrow_p^x \uparrow_p^y|. \quad (3.8)$$

Note that (3.5) and (3.8) imply that the hinge  $p \prec_x^y \subset \Delta \subset \mathbb{S}^{k-2}$  is the comparison hinge of the hinge  $\uparrow_{q_k}^p \prec_{\uparrow_{q_k}^p}^{\uparrow_{q_k}^y} \subset \Sigma_{q_k} M$ . Then by Theorem 0.4, we have  $|\uparrow_{q_k}^x \uparrow_{q_k}^y| \leq |xy|$ , which together with (3.6) implies that

$$|\uparrow_{q_k}^x \uparrow_{q_k}^y| = |xy|. \quad (3.9)$$

Note that there is a unique geodesic  $[xy]$  between any  $x \in \Delta^\circ$  and  $y \in \Delta^\circ$ . By Theorem 0.5, (3.9) implies that the triangle  $\triangle q_k xy$  composed by geodesics  $[q_k x]$ ,  $[q_k y]$  and  $[xy]$  is



isometric to its comparison triangle (in  $\mathbb{S}^2$ ). Therefore, we can conclude that  $\{q_k\} * \Delta^\circ$  can be isometrically embedded into  $M$ , so  $\{q_k\} * \Delta$  can be isometrically embedded into  $M$  (by a standard limit argument). Recall that  $\Delta = g(\Delta_+^{k-2})$ , whose vertices are  $q_1, \dots, q_{k-1}$ . It then follows that  $\Delta_+^{k-1} (= \{q_k\} * \Delta)$  can be isometrically embedded into  $M$  with  $q_1, \dots, q_k$  being the vertices.  $\square$

*Proof of the latter part of Theorem D under the assumption  $k = n + 1$ .*

We will give the proof by the induction on  $n$ .

Obviously, when  $n = 1$ ,  $M$  is either an arc of length  $\frac{\pi}{2}$  with  $q_1$  and  $q_2$  being end points or a circle of perimeter  $\pi$  with  $q_1$  and  $q_2$  being antipodal points.

Now we assume that  $n > 1$ . From the proof of the former part of Theorem D, we know that  $N \triangleq \{q_{n+1}\}^{\geq \frac{\pi}{2}} = \{q_{n+1}\}^{\frac{\pi}{2}}$  which is convex in  $M$ ; and thus  $N \in \mathcal{A}(1)$  with  $\text{diam}(N) \leq \frac{\pi}{2}$ , and  $\dim(N) \leq n - 1$  (Lemma 2.5). On the other hand, note that  $\{q_1, \dots, q_n\} \subset N$ , so by the former part of Theorem D we have  $\dim(N) = n - 1$ . Hence, by induction we can conclude that  $N$  is a glued space of finite copies of  $\Delta_+^{n-1}$  along some “faces”  $\Delta_+^{n-2}$  of them.

Given an arbitrary  $\Delta_+^{n-1} \subset N$  and any point  $p \in (\Delta_+^{n-1})^\circ$ , from the above proof for the former part, we know that a geodesic  $[q_{n+1}p]$  determines a  $\Delta_+^n$  (with  $[q_{n+1}p] \subset \Delta_+^n$  and  $q_1, \dots, q_{n+1}$  being the vertices) which can be isometrically embedded into  $M$ . As a result,  $\Sigma_p \Delta_+^n$  can be isometrically embedded into  $\Sigma_p M$  ([BGP]). Note that

$$\Sigma_p \Delta_+^n = \{\uparrow_p^{q_{n+1}}\} * \Sigma_p \Delta_+^{n-1} = \{\uparrow_p^{q_{n+1}}\} * \mathbb{S}^{n-2}$$

(which is a half  $\mathbb{S}^{n-1}$ ). This implies that if there is another geodesic  $[q_{n+1}p]'$ , then  $\Sigma_p M = \{\uparrow_p^{q_{n+1}}, (\uparrow_p^{q_{n+1}})'\} * \mathbb{S}^{n-2} = \mathbb{S}^{n-1}$ . Hence, there are at most two geodesics between  $p$  and  $q_{n+1}$ , and so there are at most two  $\Delta_+^n$  (with  $q_1, \dots, q_{n+1}$  being the vertices) which contain the given  $\Delta_+^{n-1}$ . On the other hand, of course, every  $\Delta_+^n \subset M$  with  $q_1, \dots, q_{n+1}$  being the vertices contains a  $\Delta_+^{n-1}$  in  $N$ . It follows that  $M$  contains only finite copies of  $\Delta_+^n$  because  $N$  is a glued space of finite copies of  $\Delta_+^{n-1}$ .

Now we let  $M'$  denote the union of all  $\Delta_+^n$  with  $q_1, \dots, q_{n+1}$  being the vertices.

**Claim:**  $M' = M$ . If the claim is not true, then for any  $x \in M \setminus M'$  there is  $p \in M'$  such that  $|xp| = \min_{q \in M'} \{|xq|\}$  (note that  $M'$  is compact). Obviously,  $p$  cannot be an interior point of any  $\Delta_+^n$  in  $M'$ .

**Subclaim:**  $p$  cannot be an interior point of any “face” of any  $\Delta_+^n$  in  $M'$  either. If the subclaim is not true, then we can rearrange all  $q_i$  such that  $p$  is an interior point of  $\Delta_+^{n-1} \triangleq \Delta_+^n \cap N$  for some  $\Delta_+^n$  in  $M'$ . Let  $[q_{n+1}p]$  be the geodesic between  $q_{n+1}$  and  $p$  in the  $\Delta_+^n$ . Note that  $\Sigma_p \Delta_+^{n-1} (= \mathbb{S}^{n-2})$  is convex in  $\Sigma_p M$ , and  $\Sigma_p \Delta_+^n = \{\uparrow_p^{q_{n+1}}\} * \Sigma_p \Delta_+^{n-1}$  (which is a half  $\mathbb{S}^{n-1}$ ). By Lemma 2.3, we have  $(\Sigma_p \Delta_+^{n-1})^{\geq \frac{\pi}{2}} = (\Sigma_p \Delta_+^{n-1})^{\frac{\pi}{2}}$ . Now we select a geodesic  $[xp]$ . Since  $|xp| = \min_{q \in \Delta_+^n} \{|xq|\}$ , by the first variation formula ([BGP]) we have  $|\uparrow_p^x \xi| \geq \frac{\pi}{2}$  for any  $\xi \in \Sigma_p \Delta_+^{n-1}$ , i.e.  $\uparrow_p^x \in (\Sigma_p \Delta_+^{n-1})^{\geq \frac{\pi}{2}}$ . It then follows that  $\{\uparrow_p^x, \uparrow_p^{q_{n+1}}\} \subset (\Sigma_p \Delta_+^{n-1})^{\frac{\pi}{2}}$ . On the other hand,  $(\Sigma_p \Delta_+^{n-1})^{\geq \frac{\pi}{2}}$  is convex in  $\Sigma_p M$  (Lemma 2.2); then  $\dim((\Sigma_p \Delta_+^{n-1})^{\frac{\pi}{2}}) = 0$  (Lemma 2.5), and so  $(\Sigma_p \Delta_+^{n-1})^{\frac{\pi}{2}} = \{\uparrow_p^x, \uparrow_p^{q_{n+1}}\}$  with  $|\uparrow_p^x \uparrow_p^{q_{n+1}}| = \pi$ . By the first variation formula ([BGP]),

there exists another geodesic  $[pq_{n+1}]'$  in  $M$  such that  $|\uparrow_p^x (\uparrow_p^{q_{n+1}})'| \leq \frac{\pi}{2}$  (otherwise there is  $x' \in [px]$  such that  $|q_{n+1}x'| > \frac{\pi}{2}$  which contradicts  $\text{diam}(M) \leq \frac{\pi}{2}$ ). However, from the above,  $[pq_{n+1}]'$  determines another  $\Delta_+^n$  with  $q_1, \dots, q_{n+1}$  being the vertices and  $[pq_{n+1}]' \subset \Delta_+^n$ , and thus  $(\uparrow_p^{q_{n+1}})' \in (\Sigma_p \Delta_+^{n-1})^{\frac{\pi}{2}}$ . Hence,  $(\uparrow_p^{q_{n+1}})'$  has to be  $\uparrow_p^x$ , and so  $[px] \subset [pq_{n+1}]' \subset \Delta_+^n$  which contradicts  $x \in M \setminus M'$  (i.e., the subclaim is verified).

For convenience, a “face”  $\Delta_+^{n-1}$  of any  $\Delta_+^n \subseteq M'$  (resp. a “face”  $\Delta_+^{n-2}$  of such a  $\Delta_+^{n-1}$ ) is said to be an  $(n-1)$ -face (resp.  $(n-2)$ -face) of  $M'$ . According to the subclaim and its proof, it is not hard to observe that:

(3.10) *If  $M \neq M'$ , then for any  $x \in M \setminus M'$  and any interior point  $q$  of any  $\Delta_+^n$  in  $M'$ , the nearest point in  $[xq] \cap M'$  to  $x$  for any  $[xq]$  has to lie in an  $(n-2)$ -face of  $M'$ .*

However, by the induction on  $n$ , we will derive a contradiction under (3.10). If  $n = 2$ , note that (3.10) implies that there must be a geodesic which branches at some 0-face (which is a point) of  $M'$ , which is impossible ([BGP]). Now we assume that  $n > 2$ . Let  $v$  be the nearest point to  $x$  on  $[xq] \cap M'$  (where  $[xq]$  is the geodesic in (3.10)). Note that  $\Sigma_v M'$  is also the union of finite copies of  $\Delta_+^{n-1}$  and each  $\Delta_+^{n-1}$  can be isometrically embedded into  $\Sigma_v M$  (because each  $\Delta_+^n$  of  $M'$  can be isometrically embedded into  $M$ ); and note that  $\Sigma_v M \neq \Sigma_v M'$  because  $\uparrow_v^x \notin \Sigma_v M'$  (note that  $[vx] \cap M' = \{v\}$ ). Then from the definition of the angle (i.e. the distance between two directions in  $\Sigma_v M$ ) ([BGP]), it is not hard to see that (3.10) implies that:

(3.11) *For any  $\xi \in \Sigma_v M \setminus \Sigma_v M'$  and any interior point  $\eta$  of any  $\Delta_+^{n-1}$  in  $\Sigma_v M'$ , the nearest point in  $[\xi\eta] \cap \Sigma_v M'$  to  $\xi$  for any  $[\xi\eta]$  has to lie in an  $(n-3)$ -face of  $\Sigma_v M'$ .*

By induction, we can derive a contradiction under (3.11). That is, we get a contradiction under (3.10), so we have  $M = M'$  (i.e. the claim is verified).

Now we can conclude that  $M$  is not only the union of all  $\Delta_+^n$  with  $q_1, \dots, q_{n+1}$  being the vertices by the claim, but also a glued space of these  $\Delta_+^n$  along some  $(n-1)$ -faces (not  $(n-k)$ -faces with  $k \geq 2$ ) of them from the proof of the claim.  $\square$

**Remark 3.12.** In fact, if  $k = n + 1$  in Theorem D, then we can determine all the possible structures of  $M$  by the induction on the dimension  $n$ . From the above proof, we know that  $N \triangleq \{q_{n+1}\}^{\frac{\pi}{2}}$  belongs to  $\mathcal{A}^{n-1}(1)$  (because  $N$  is convex in  $M$ ) with  $\text{diam}(N) \leq \frac{\pi}{2}$  and  $\{q_1, \dots, q_n\} \subset N$ ; so by induction we can determine all the possible structures of  $N$ . Moreover, we know that there are at most two geodesics between  $q_{n+1}$  and any interior point of any  $\Delta_+^{n-1}$  in  $N$ . On the other hand,  $M$  is a glued space of all  $\Delta_+^n$  along some  $(n-1)$ -faces of them. Hence, there are at most two geodesics between  $q_{n+1}$  and any point in  $N$ ; and for any  $x \in M$ , there is some geodesic  $[q_{n+1}p]$  with  $p \in N$  such that  $x \in [q_{n+1}p]$ . Then we can prove that either  $M = \{q_{n+1}\} * N$ , or  $\tilde{N} \triangleq \Sigma_{q_{n+1}} M$  admits an isometry  $\sigma$  of order 2 (i.e.  $\sigma^2 = \text{id}$ ), which naturally induces an isometry  $\tilde{\sigma}$  of order 2 on the suspension  $\{q_{n+1}, \bar{q}_{n+1}\} * \tilde{N}$  (where  $|q_{n+1}\bar{q}_{n+1}| = \pi$ ) with  $\tilde{\sigma}([q_{n+1}p]) = [\bar{q}_{n+1}\sigma(p)]$  and  $\tilde{\sigma}([\bar{q}_{n+1}p]) = [q_{n+1}\sigma(p)]$ , such that  $N = \tilde{N}/\langle\sigma\rangle$  and  $M = (\{q_{n+1}, \bar{q}_{n+1}\} * \tilde{N})/\langle\tilde{\sigma}\rangle$  (one can give the detailed proof for this by referring to the proof of Lemma 2.5 in [SSW], or [RW]). This implies that we can determine the

structure of  $M$  by that of  $N$ . For example, we give all the possible structures of  $M$  for  $n = 1, 2$  and  $k = n + 1$  (for convenience, we let  $\mathbb{Z}_2$  denote both  $\langle \sigma \rangle$  and  $\langle \tilde{\sigma} \rangle$ ).

$n = 1$  and  $k = 2$ :  $M = \Delta_+^1 (= [q_1 q_2])$ , an arc of length  $\frac{\pi}{2}$ ; or  $M = S_\pi^1$ , a circle of perimeter  $\pi$ , which is a glued space of two copies of  $\Delta_+^1$  at  $q_1$  and  $q_2$ .

$n = 2$  and  $k = 3$ :  $M = \Delta_+^2 (= \{q_3\} * [q_1 q_2])$ ,  $\{q_3\} * S_\pi^1$  (a glued space of two  $\Delta_+^2$  along  $[q_3 q_1]$  and  $[q_3 q_2]$ ),  $(\{q_3, \bar{q}_3\} * S_\pi^1)/\mathbb{Z}_2$  (a glued space of two  $\Delta_+^2$  along  $[q_3 q_1]$ ,  $[q_3 q_2]$  and  $[q_1 q_2]$ ) where  $\mathbb{Z}_2$  acts on  $S_\pi^1$  by a reflection (note that  $S_\pi^1/\mathbb{Z}_2 = [q_1 q_2]$ ), or  $(\{q_3, \bar{q}_3\} * S_{2\pi}^1)/\mathbb{Z}_2 (= \mathbb{RP}^2)$ , a glued space of four  $\Delta_+^2$  along their boundaries) where  $\mathbb{Z}_2$  acts on  $S_{2\pi}^1$  by the antipodal map (note that  $S_{2\pi}^1/\mathbb{Z}_2 = S_\pi^1$ ).

We will end this section by giving a brief proof for Corollary E.

#### Proof of Corollary E:

Note that  $M$  has empty boundary (because  $M$  is closed). According to Remark 3.12, it has to hold that  $\tilde{N} (= \Sigma_{q_{n+1}} M)$  admits an isometrical  $\mathbb{Z}_2$ -action which naturally induces an isometrical  $\mathbb{Z}_2$ -action on the suspension  $\{q_{n+1}, \bar{q}_{n+1}\} * \tilde{N}$  such that  $M = (\{q_{n+1}, \bar{q}_{n+1}\} * \tilde{N})/\mathbb{Z}_2$ . Since  $M$  is a Riemannian manifold, we have  $\tilde{N} = \Sigma_{q_{n+1}} M = \mathbb{S}^{n-1}$ ; and thus  $\{q_{n+1}, \bar{q}_{n+1}\} * \tilde{N} = \mathbb{S}^n$ , so  $M = \mathbb{S}^n/\mathbb{Z}_2$ . Since  $\mathbb{Z}_2$  acts on  $\mathbb{S}^n$  by isometries and  $M$  is a Riemannian manifold (with  $\text{diam}(M) \leq \frac{\pi}{2}$ ), it has to hold that  $M = \mathbb{RP}^n$  (i.e. the  $\mathbb{Z}_2$ -action on  $\mathbb{S}^n$  must be realized by the antipodal map).  $\square$

## 4 A technical corollary of Theorem D

We first give an easy corollary of Theorem D.

**Proposition 4.1.** *Let  $M \in \mathcal{A}^n(1)$ , and let  $\{p_1, \dots, p_k\}$  be a  $\frac{\pi}{2}$ -separated subset in  $M$  with  $|p_1 p_i| > \frac{\pi}{2}$  for  $2 \leq i \leq k$ . Suppose that  $N$  is a complete and convex subset in  $\{p_1, \dots, p_k\}^{\geq \frac{\pi}{2}}$  with  $\text{diam}(N) \leq \frac{\pi}{2}$ , and that  $\{q_1, \dots, q_h\}$  is a  $\frac{\pi}{2}$ -separated subset in  $N$ . Then  $k + h \leq n + 2$ .*

*Proof.* Note that  $N \in \mathcal{A}(1)$  (because  $N$  is convex in  $\{p_1, \dots, p_k\}^{\geq \frac{\pi}{2}}$  which is convex in  $M$  by Lemma 2.2) with  $\text{diam}(N) \leq \frac{\pi}{2}$ . By Theorem D,  $\Delta_+^{h-1}$  can be isometrically embedded into  $N$  with  $q_1, \dots, q_h$  being the vertices. Now we consider  $\Sigma_{q_h} M \in \mathcal{A}^{n-1}(1)$ . Similar to the proof of Theorem A, we can conclude that any  $\{\uparrow_{q_h}^{p_1}, \dots, \uparrow_{q_h}^{p_k}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_h} M$  with  $|\uparrow_{q_h}^{p_1} \uparrow_{q_h}^{p_i}| > \frac{\pi}{2}$  for  $2 \leq i \leq k$ . Moreover,  $\Sigma_{q_h} \Delta_+^{h-1} = \Delta_+^{h-2}$ , which can be isometrically embedded into  $\{\uparrow_{q_h}^{p_1}, \dots, \uparrow_{q_h}^{p_k}\}^{\geq \frac{\pi}{2}}$  (with some  $\uparrow_{q_h}^{q_1}, \dots, \uparrow_{q_h}^{q_{h-1}}$  being the vertices). Then we can repeat such an argument on  $\Sigma_{\uparrow_{q_h}^{q_{h-1}}}(\Sigma_{q_h} M)$ , so by induction (note that when  $h = 0$ , the proposition is obvious by Theorem A) we obtain that  $k + h - 1 \leq n - 1 + 2$ , i.e.  $k + h \leq n + 2$ .  $\square$

Based on Proposition 4.1, we give a technical corollary which may make sense in analyzing the direction spaces as same as in [P1].

**Corollary 4.2. (A technical corollary of Theorem D)** *Let  $M, \{p_1, \dots, p_k\}$  and  $N$  be the same as in Proposition 4.1. Then given any  $z_1 \in N$  and sufficiently small  $\epsilon > 0$ , there exist  $\{z_2, \dots, z_h\} \subset \partial B(z_1, \epsilon) \cap N$  with  $h \leq n + 2 - k$  and  $|z_i z_j| > \epsilon$  ( $2 \leq i \neq j \leq h$ ) and  $\delta_1, \delta_2 > 0$  with  $h\delta_2 < \delta_1$  such that for any  $p \in M$  either  $|pN| > \delta_2$  or there is some  $1 \leq i \leq h$  such that  $|pz_i| < \frac{\pi}{2} - \delta_1$  and  $|pz_{i'}| < \frac{\pi}{2} + \delta_2$  for all  $i' \neq i$ .*

*Proof.* Note that we can find a maximal  $\frac{\pi}{2}$ -separated subset in  $N$ ,  $\{z_1, q_2, \dots, q_h\}$  (i.e.  $|x\{z_1, q_2, \dots, q_h\}| < \frac{\pi}{2}$  for all  $x \in N$ ). By Proposition 4.1, we have  $h \leq n + 2 - k$ . In  $N$ , we consider  $L \triangleq \{z_1\}^{\geq \frac{\pi}{2}}$ , which is convex in  $N$  (so in  $M$ ) by Lemma 2.2 (so  $L \in \mathcal{A}(1)$ ). Since  $\text{diam}(N) \leq \frac{\pi}{2}$ , we have  $L = \{z_1\}^{=\frac{\pi}{2}}$  and  $\{q_2, \dots, q_h\} \subset L$ .

Next we will find  $z_i$  for  $2 \leq i \leq h$ . Note that we can select  $\bar{q}_i \in L^\circ$  (the interior part of  $L$ ) such that  $|\bar{q}_i q_i| \ll \epsilon$  ([BGP]) (if  $q_i \in L^\circ$ , then we let  $\bar{q}_i$  just be  $q_i$ ). Then we select an arbitrary geodesic  $[z_1 \bar{q}_i]$ , and select  $z_i \in [z_1 \bar{q}_i]$  with  $|z_1 z_i| = \epsilon$ . Of course,  $\{z_2, \dots, z_h\} \subset \partial B(z_1, \epsilon) \cap N$ . By Theorem 0.4 on the triangle  $\triangle z_1 \bar{q}_i \bar{q}_j$ , it is not hard to see that  $|z_i z_j| > \epsilon$  for  $2 \leq i \neq j \leq h$  (note that  $|z_1 \bar{q}_i| = |z_1 \bar{q}_j| = \frac{\pi}{2}$  and  $|\bar{q}_i \bar{q}_j|$  is almost equal to  $\frac{\pi}{2}$ , and  $z_i \in [z_1 \bar{q}_i], z_j \in [z_1 \bar{q}_j]$ ).

**Claim:** *There exists  $\delta > 0$  such that  $|x\{z_1, \dots, z_h\}| < \frac{\pi}{2} - \delta$  for any  $x \in N$  (i.e. there is some  $1 \leq i \leq h$  such that  $|xz_i| < \frac{\pi}{2} - \delta$ ).*

It is easy to see that the claim implies that we can find the desired  $\delta_1$  and  $\delta_2$ , so we only need to verify the claim in the rest of proof.

Since  $\{z_1, q_2, \dots, q_k\}$  is a maximal  $\frac{\pi}{2}$ -separated subset in  $N$  and  $|\bar{q}_i q_i| \ll \epsilon$ , there is  $\epsilon_1 > 0$  such that  $|y\{\bar{q}_2, \dots, \bar{q}_k\}| < \frac{\pi}{2} - \epsilon_1$  for any  $y \in L$ , i.e. there is a  $\bar{q}_i$  such that  $|y \bar{q}_i| < \frac{\pi}{2} - \epsilon_1$ . On the other hand, by the first variation formula ([BGP]) we have  $|\uparrow_{\bar{q}_i}^{z_1} \xi| \geq \frac{\pi}{2}$  for any  $\xi \in \Sigma_{\bar{q}_i} L$  (note that  $L = \{z_1\}^{=\frac{\pi}{2}}$  which is convex in  $N$ ). In fact, by Lemma 2.3 we have  $|\uparrow_{\bar{q}_i}^{z_1} \xi| = \frac{\pi}{2}$  because  $\Sigma_{\bar{q}_i} L$  is convex in  $\Sigma_{\bar{q}_i} N$  and  $\Sigma_{\bar{q}_i} L$  has empty boundary (note that  $L$  is convex in  $N$  and  $\bar{q}_i \in L^\circ$ ). It then follows that  $|\uparrow_{\bar{q}_i}^{z_i} \uparrow_{\bar{q}_i}^y| = \frac{\pi}{2}$  for any geodesic  $[\bar{q}_i y] \subset L$ . And thus by Theorem 0.4 on the hinge  $\bar{q}_i \prec_y^{z_i}$ , there is  $\chi(\epsilon, \epsilon_1) > 0$  (where  $\chi(\epsilon, \epsilon_1) \rightarrow 0$  as  $\epsilon, \epsilon_1 \rightarrow 0$ ) such that

$$|yz_i| < \frac{\pi}{2} - \chi(\epsilon, \epsilon_1)$$

(note that  $|\bar{q}_i y| < \frac{\pi}{2} - \epsilon_1$ ,  $|\bar{q}_i z_i| = \frac{\pi}{2} - \epsilon$  and  $|\uparrow_{\bar{q}_i}^{z_i} \uparrow_{\bar{q}_i}^y| = \frac{\pi}{2}$ ). Hence, for any  $x \in B(L, \epsilon_2) \cap N$  (the  $\epsilon_2$ -tubular neighborhood of  $L$  in  $N$ ) there exist a  $z_i$  and  $\chi(\epsilon, \epsilon_1, \epsilon_2) > 0$  such that

$$|xz_i| < \frac{\pi}{2} - \chi(\epsilon, \epsilon_1, \epsilon_2).$$

On the other hand, there is a  $\epsilon_3 > 0$  such that  $|xz_1| < \frac{\pi}{2} - \epsilon_3$  for any  $x \in N \setminus B(L, \epsilon_2)$  (otherwise there is  $x_0 \in N \setminus B(L, \epsilon_2)$  such that  $|x_0 z_1| = \frac{\pi}{2}$ , i.e.  $x_0 \in L$ ; a contradiction). Note that  $\delta \triangleq \min\{\chi(\epsilon, \epsilon_1, \epsilon_2), \epsilon_3\}$  is the desired number of the claim.  $\square$

## Appendix

### Proof of Theorem 0.2:

We will give the proof by the induction on the dimension  $n$ . Obviously, Theorem 0.2 is true if  $n = 0$  and  $1$ . Now we assume  $n > 1$ . We consider  $\Sigma_{q_1}M$  which belongs to  $\mathcal{A}^{n-1}(1)$  ([BGP]). By Lemma 1.2, it is easy to see that in  $\{q_2, \dots, q_k\}$  there is at most one point, say  $q_k$ , such that  $|q_1 q_k| = \pi$ . Then by Lemma 1.1, any  $\{\uparrow_{q_1}^{q_2}, \dots, \uparrow_{q_1}^{q_{k-1}}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $\Sigma_{q_1}M$ . By the inductive assumption on  $\Sigma_{q_1}M$ , we have

$$k - 2 \leq 2(n - 1 + 1), \text{ i.e., } k \leq 2(n + 1).$$

Moreover, if  $k = 2(n + 1)$ , then it has to hold that  $|q_1 q_{2(n+1)}| = \pi$ ; and so  $M = \{q_1, q_{2(n+1)}\} * M_1$  for some  $M_1 \in \mathcal{A}^{n-1}(1)$  (Lemma 1.2). This implies that  $\{q_2, \dots, q_{2(n+1)}\}$  is a  $\frac{\pi}{2}$ -separated subset in  $M_1$ . By the inductive assumption on  $M_1$ , we have that  $M_1$  is isometric to the unit sphere  $\mathbb{S}^{n-1}$ . Therefore, we can conclude that  $k = 2(n + 1)$  if and only if  $M$  is isometric to  $\{q_1, q_{2(n+1)}\} * \mathbb{S}^{n-1}$  which is the unit sphere  $\mathbb{S}^n$ .  $\square$

**Proof of Lemma 1.4:**

We will give the proof by the induction on  $n$ . Obviously, if  $n = 1$ , the lemma is true (because  $M$  is an arc of length  $\leq \pi$  if  $M \in \mathcal{A}^1(1)$  has nonempty boundary (see the convention after Lemma 1.2)). Now we assume that  $n > 1$ .

Since  $\partial M$  is compact (because  $\partial M$  is closed in  $M$  ([BGP]) and  $M$  is compact), we select  $q \in \partial M$  with  $|pq| = |p\partial M|$  and a geodesic  $[pq]$ . By the first variation formula,

$$|\uparrow_q^p \xi| \geq \frac{\pi}{2}$$

in  $\Sigma_q M \in \mathcal{A}^{n-1}(1)$  for any  $\xi \in \partial(\Sigma_q M)$  (refer to [BGP] for the details on the boundary in Alexandrov geometry). By induction, we have  $\Sigma_q M = \{\uparrow_q^p\} * \partial(\Sigma_q M)$ , so  $|\uparrow_q^p \eta| \leq \frac{\pi}{2}$  for any  $\eta \in \Sigma_q M$  and the “=” holds if and only if  $\eta \in \partial(\Sigma_q M)$ . Then by Theorem 0.4 on any triangle  $\triangle pqr$  with  $r \in \partial M$  (note that  $|pr| \geq |pq| \geq \frac{\pi}{2}$  and  $\angle pqr \leq \frac{\pi}{2}$ ), it has to hold that

$$|pr| = |pq| = \frac{\pi}{2} \text{ and } \angle pqr = \frac{\pi}{2}.$$

It then follows that  $|pr| = |p\partial M| = \frac{\pi}{2}$ . And  $\uparrow_q^r \in \partial(\Sigma_q M)$  for any geodesic  $[qr]$ , so  $[qr]$  belongs to  $\partial M$  ([BGP]). Now we can take the place of  $q$  by any  $r \in \partial M$ . Similarly, we can get that any geodesic  $[rr']$  for any other  $r' \in \partial M$  belongs to  $\partial M$ , i.e.  $\partial M$  is convex in  $M$ ; and  $\Sigma_r M = \{\uparrow_r^p\} * \partial(\Sigma_r M)$ , which implies that there is a unique geodesic between  $p$  and  $r$ . Hence, in order to prove that  $M = \{p\} * \partial M$ , it suffices to show that there is a point  $r$  in  $\partial M$  such that  $x \in [pr]$  for any  $x \in M$  (cf. Proposition 2.8 and refer to [SSW]). Similarly, we can select  $s \in \partial M$  such that  $|xs| = |x\partial M|$  and a geodesic  $[xs]$ ; and get that  $|\uparrow_s^x \xi| \geq \frac{\pi}{2}$  in  $\Sigma_s M$  for any  $\xi \in \partial(\Sigma_s M)$ , and so  $\Sigma_s M = \{\uparrow_s^x\} * \partial(\Sigma_s M)$ . It then has to hold that

$$\uparrow_s^x = \uparrow_s^p.$$

On the other hand, note that we have proved that  $|x\partial M| \leq \frac{\pi}{2}$ . It then follows that  $[xs] \subseteq [ps]$  (of course  $x \in [ps]$ ).  $\square$

**Proof of Lemma 2.11:**

Due to Proposition 2.8, it suffices to show that there is a unique geodesic between any  $x \in X$  and  $y \in S^1$ . Note that  $\Sigma_x X$  is a complete convex subset in  $\Sigma_x M$  and  $\Sigma_x X$  has empty boundary because  $X$  is a complete convex subset in  $M$  and  $X$  has empty boundary. By Lemma 2.2,  $(\Sigma_x X)^{\geq \frac{\pi}{2}}$  is convex in  $\Sigma_x M$ ; and by Lemma 2.3, we have  $(\Sigma_x X)^{\geq \frac{\pi}{2}} = (\Sigma_x X)^{= \frac{\pi}{2}}$ . It then follows from Lemma 2.5 that  $\dim((\Sigma_x X)^{= \frac{\pi}{2}}) \leq 1$  (note that  $\dim(\Sigma_x X) = n - 3$  and  $\dim(\Sigma_x M) = n - 1$ ). Moreover, for any  $y \in S^1$  and any geodesic  $[xy]$ , by the first variation formula we have that  $\uparrow_x^y$  belongs to  $(\Sigma_x X)^{\geq \frac{\pi}{2}}$  ( $= (\Sigma_x X)^{= \frac{\pi}{2}}$ ). This implies that  $\dim((\Sigma_x X)^{= \frac{\pi}{2}}) = 1$ . On the other hand, since  $S^1 \subset X^{= \frac{\pi}{2}}$  and  $S^1$  is convex in  $M$ , by Theorem 0.5, for any geodesic  $[y'y''] \subset S^1$  with  $y \in [y'y'']$  there is a triangle  $\triangle xy'y''$  containing  $[y'y'']$  which is isometric to its comparison triangle; moreover,  $[xy]$  belongs to the convex domain bounded by  $\triangle xy'y''$  ([GM]). Then we can define a multi-value map

$$f : S^1 \rightarrow (\Sigma_x X)^{= \frac{\pi}{2}} \text{ by } y \mapsto \{\text{all directions from } x \text{ to } y\}$$

such that for any  $y \in S^1$  and any geodesic  $[xy]$  there are a neighborhood  $U$  of  $y$  in  $S^1$  and a neighborhood  $\tilde{U}$  of  $\uparrow_x^y$  in  $(\Sigma_x X)^{= \frac{\pi}{2}}$  such that  $f|_U : U \rightarrow \tilde{U}$  is an isometry. This implies that  $f(S^1)$  is both open and closed in  $(\Sigma_x X)^{= \frac{\pi}{2}}$ , so we have  $f(S^1) = (\Sigma_x X)^{= \frac{\pi}{2}}$ . It therefore follows that  $f^{-1}$  is a covering map. Since  $(\Sigma_x X)^{= \frac{\pi}{2}} \in \mathcal{A}^1(1)$  and  $S^1$  has perimeter  $> \pi$ , it has to hold that  $f^{-1}$  is a 1-1 covering map (of course  $f$  is also a 1-1 map), which implies that there is a unique geodesic between  $x \in X$  and  $y \in S^1$ .  $\square$

**Remark A.1.** According to the proof of Lemma 2.11, if  $X$  has nonempty boundary in Lemma 2.11, we can still conclude that there is a unique geodesic between any interior point  $x$  in  $X$  and  $y \in S^1$ ; as a result,  $X * S^1$  can be isometrically embedded into  $M$  (cf. Proposition 2.8 and refer to [SSW]).

**Remark A.2.** In fact, Lemma 2.11 has the following generalized version: *Let  $X$  and  $Y$  be two complete convex subsets in  $M \in \mathcal{A}^n(1)$ . Then  $M = X * Y$  if (i)  $\dim(X) + \dim(Y) = n - 1$ ; (ii) both  $X$  and  $Y$  have empty boundary; (iii) either  $X$  or  $Y$  has radius  $> \frac{\pi}{2}$ ; (iv)  $|xy| = \frac{\pi}{2}$  for any  $x \in X$  and  $y \in Y$ .*

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